Extended formulations for some 0-1 symmetry-breaking polytopes

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Outline

- Motivation: symmetric 0-1 formulations
- Partitioning Orbitopes in the original and in an extended space
- Applications
- More Orbitopes

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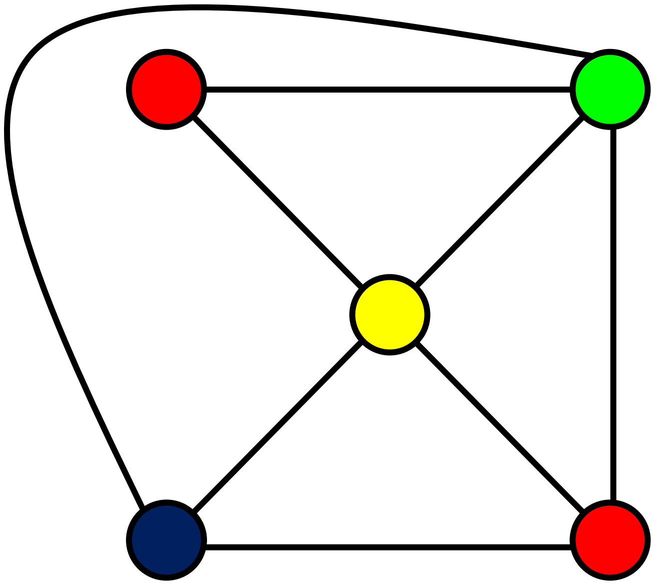
The graph coloring problem



Input: a connected graph G (V,E)

k-coloring: a mapping G $\longrightarrow$ {1,…, k}; such that no pair of adjacent vertices have the same color;

Graph coloring problem: find minimum k such that a k-coloring exists.

Graph coloring: a 0-1 formulation

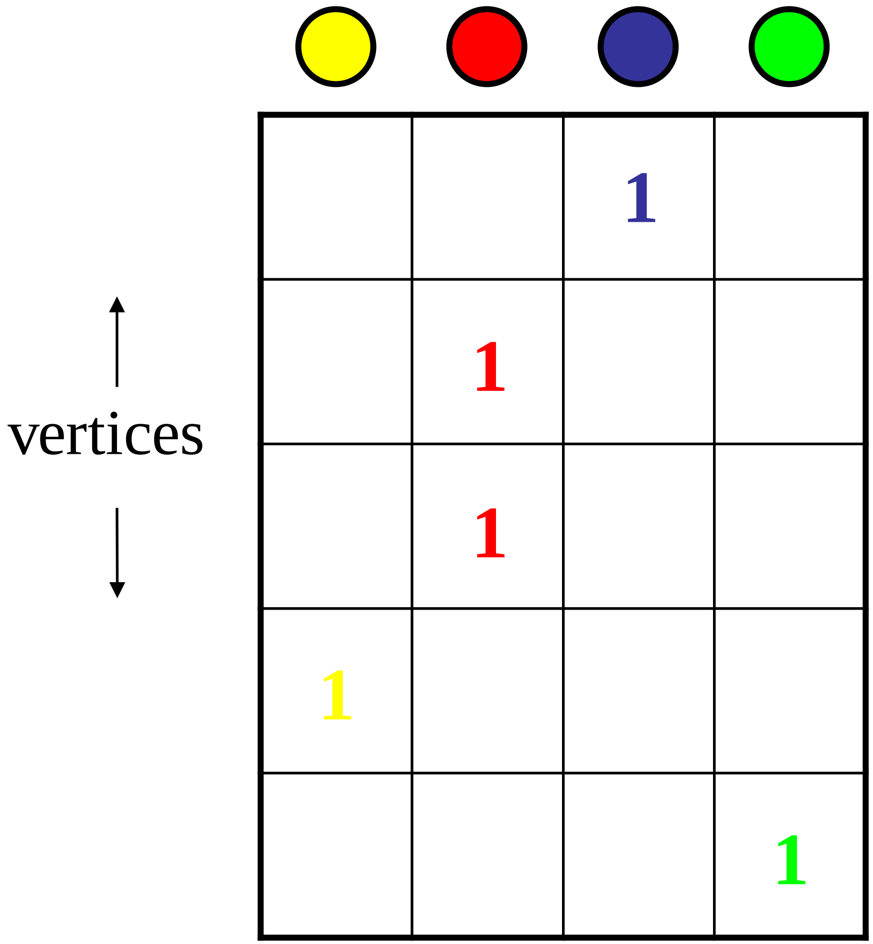
$$x_{ij} = \begin{cases} 1 & \text{if color } j \text{ is given to vertex i} \\ 0 & otherwise \end{cases}$$

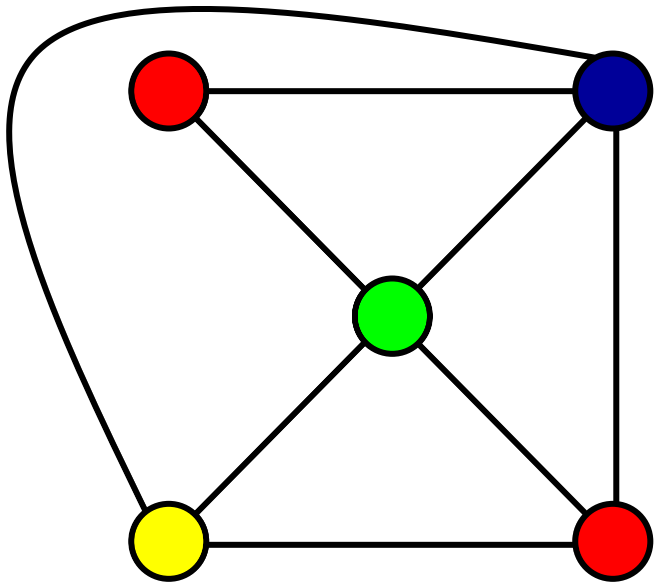
$$y_j = \begin{cases} 1 & \text{if } x_{ij} = 1 \text{ for some } i \in V \\ 0 & \text{otherwise} \end{cases}$$

Graph coloring: a 0-1 formulation

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{|S|} y_j \\
\text{s.t.} \quad & x_{ij} + x_{kj} \le y_j, \quad \{i,k\} \in E,\ j \in S \\
& \sum_{j=1}^{|S|} x_{ij} = 1, \qquad i \in V \\
& x_{ij} \in \{0,1\} \qquad i \in V, j \in S \\
& y_j \in \{0,1\} \qquad j \in S
\end{aligned}
$$

Symmetry harms: symmetric groups(1)

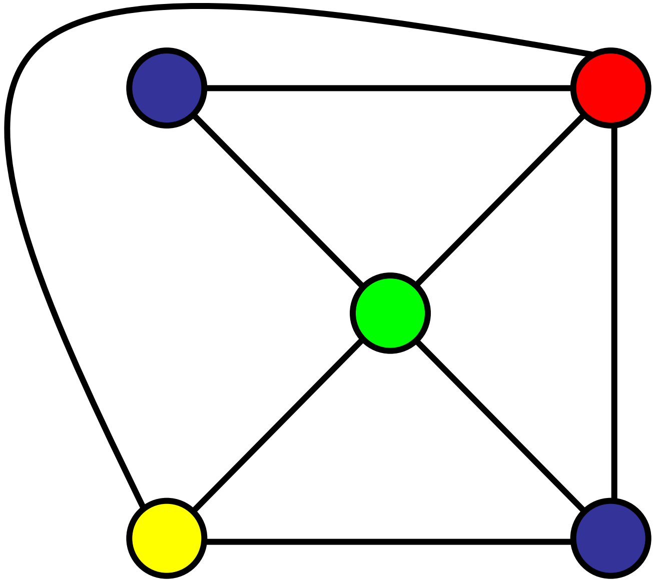




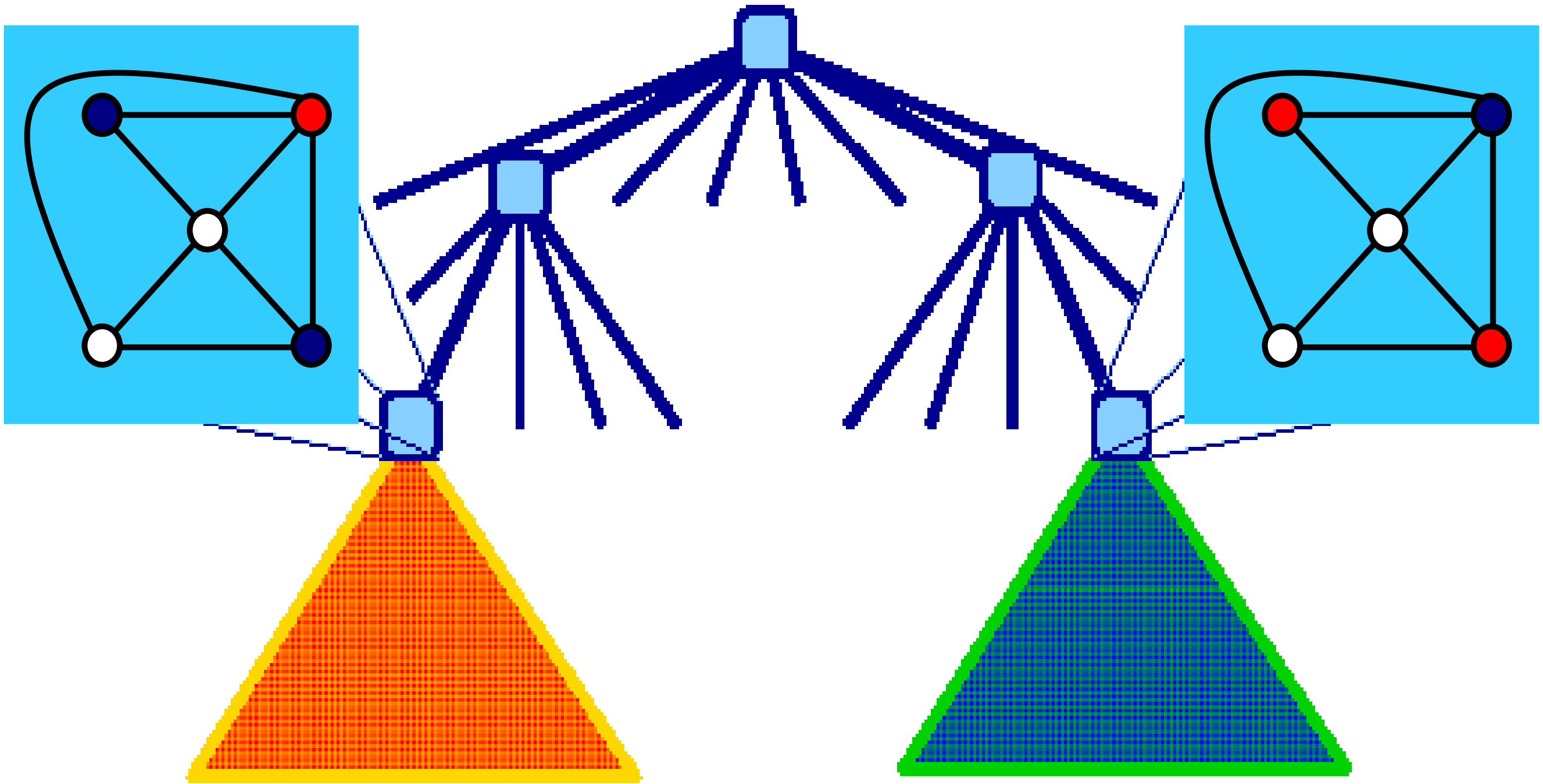
Symmetry harms: symmetric groups(1)

vertices

	1		
		1	
		1	
1			
			1



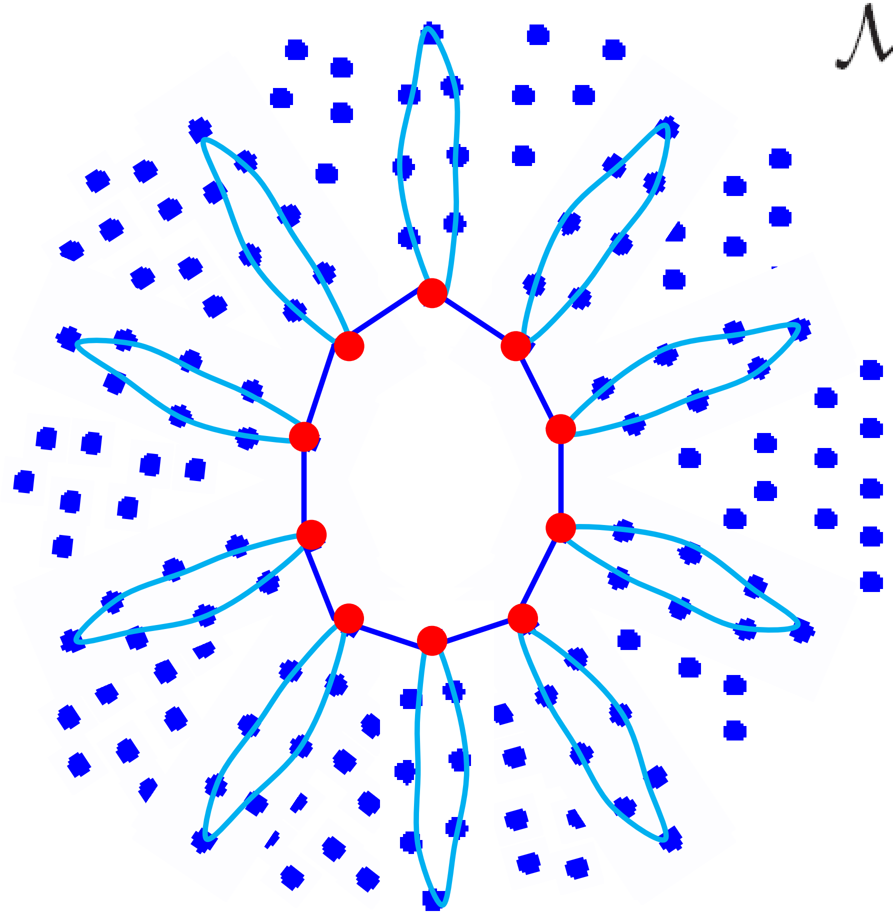
Symmetry harms: symmetric groups(2)



How to cope with symmetry

- Isomorphism pruning [Margot&al., 02+]
- Orbital branching [Linderoth&al., 07+]
- Orbitopes [06+]

Partitioning Orbitopes

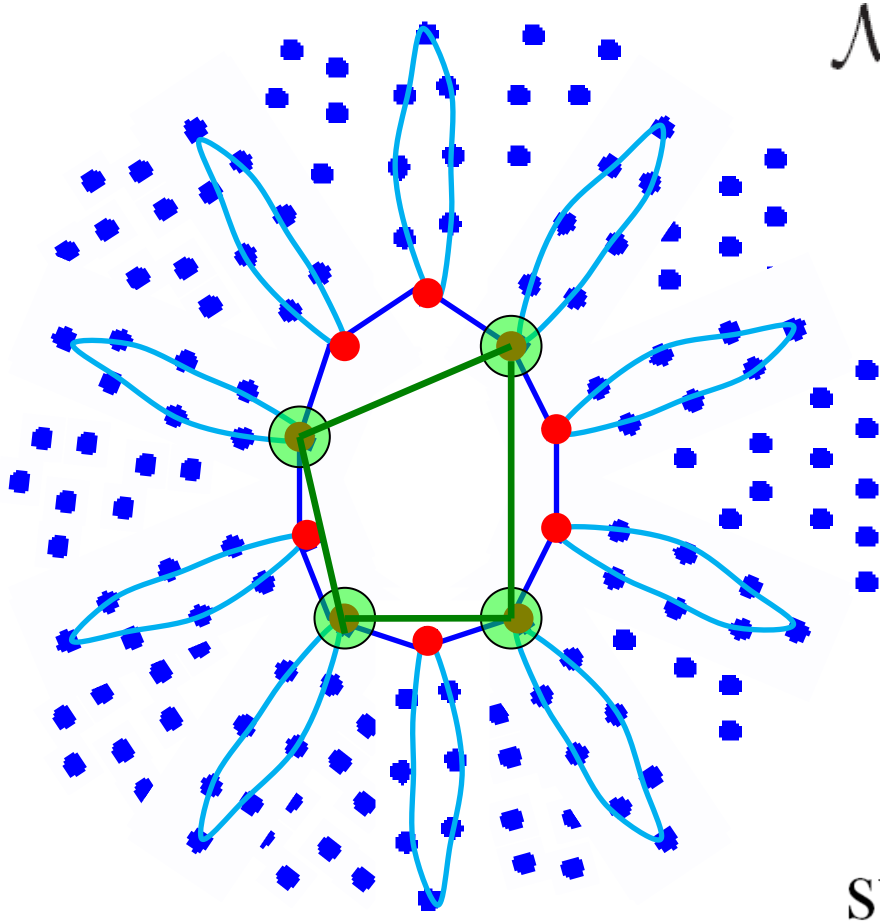


$$\mathcal{M}_{p,q} := \{0,1\}^{[p]\times[q]}$$

$\mathcal{M}^{=}_{p,q}$ $\quad G \quad$ $\mathcal{M}^{\max}_{p,q}$

$$\mathrm{O}^{=}_{p,q}(G)$$

Partitioning Orbitopes



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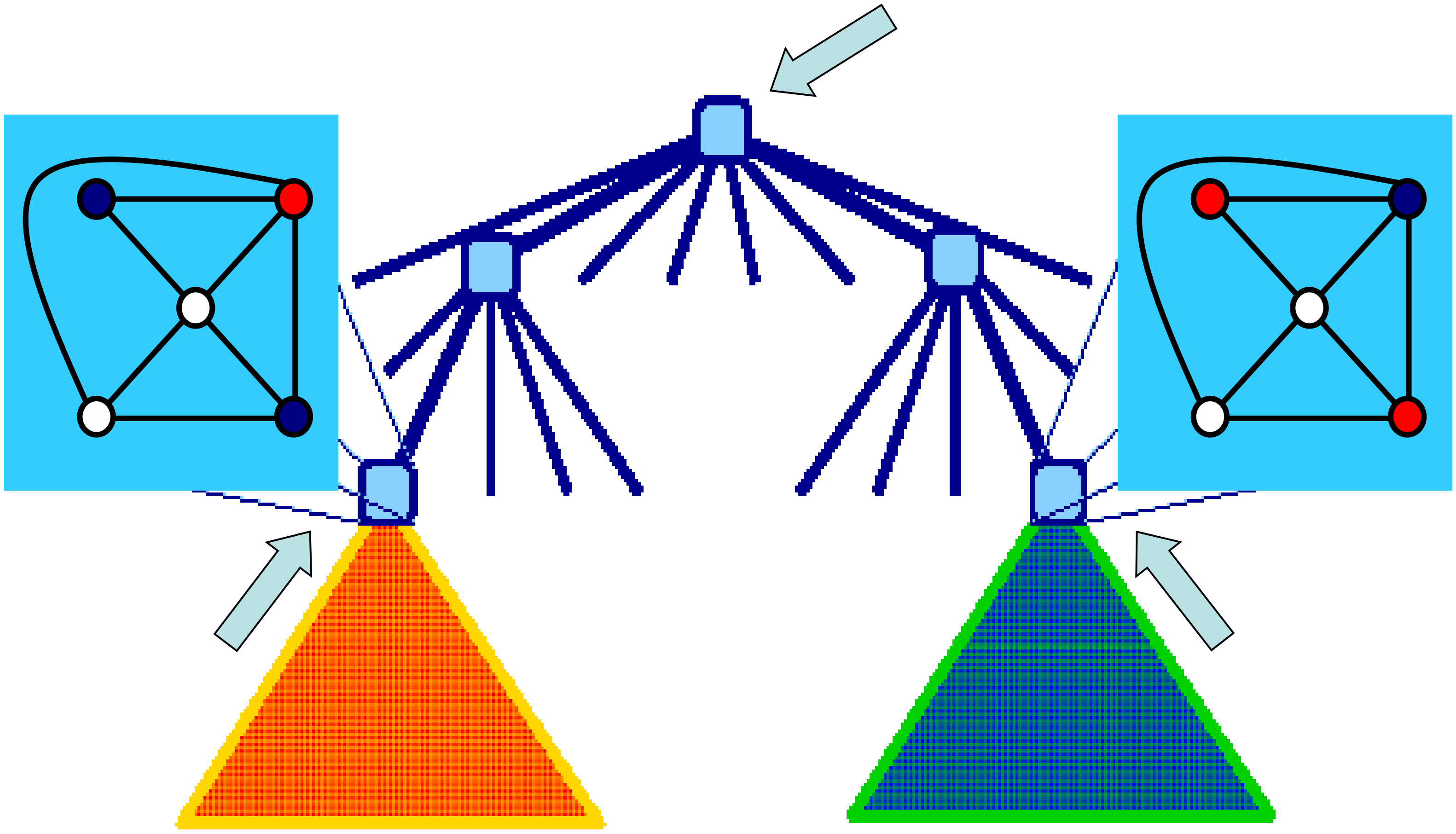
$\mathcal{M}^{=}_{p,q}$ G $\mathcal{M}^{\max}_{p,q}$

$$\mathrm{O}^{=}_{p,q}(G)$$

$$\mathrm{O}^{=}_{p,q}(G) \cap P$$

symmetry-free formulation

Plugging the Orbitopes in the branching tree



Some comments on the definition

- $O^{=}_{p,q}(G)$ breaks symmetry in all 0-1 problems with exactly one non-zero entry per row

$$\mathcal{M}_{p,q} \Rightarrow \mathcal{M}^{=}_{p,q} \underset{G}{\Rightarrow} \mathcal{M}^{\max}_{p,q} \Rightarrow O^{=}_{p,q}(G)$$

- Other Orbitopes

$$O^{\leq}_{p,q}(G) \qquad O^{\geq}_{p,q}(G) \qquad O_{p,q}(G)$$

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An example and two observations

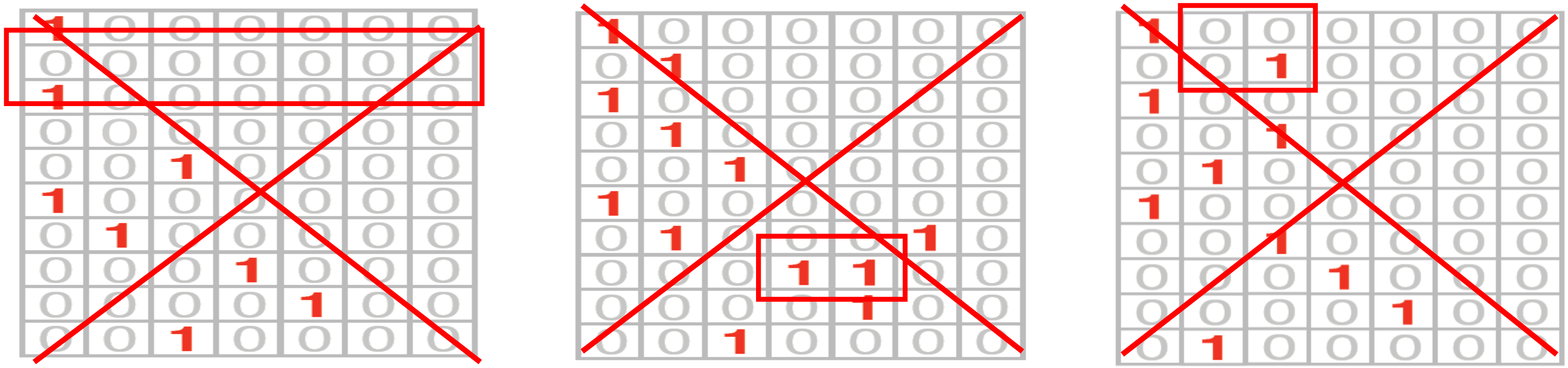
1	0	0	0	0	0	0
0	0	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	1	0	0	0	0

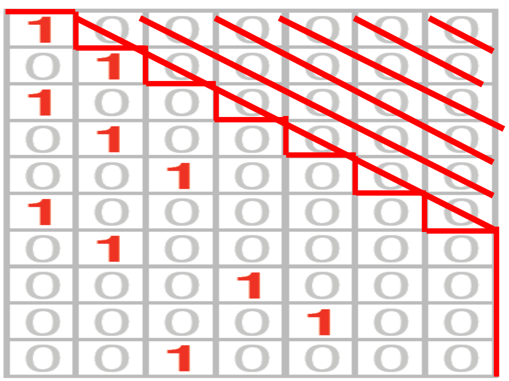
1	0	0	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	1	0
0	0	0	1	1	0	0
0	0	0	0	1	0	0
0	0	1	0	0	0	0

1	0	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	1	0	0	0	0	0

1	0	0	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	1	0	0	0	0

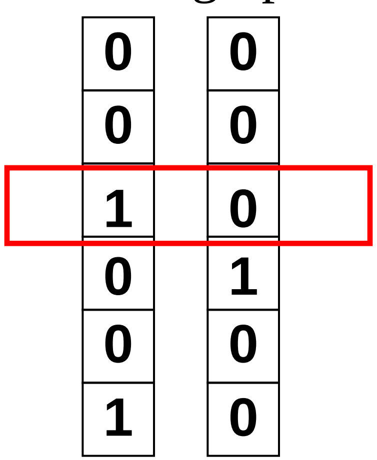
An example and two observations



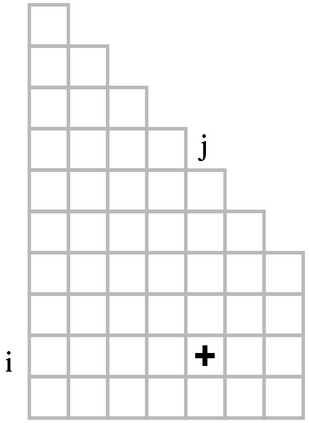


The top right can be fixed to 0

Columns come in lexicographically decreasing order

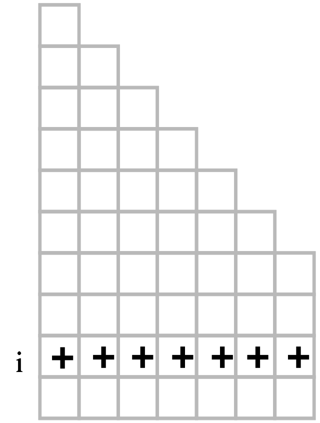


Linear constraints for $\mathrm{O}^{=}_{p,q}$ *(1)*



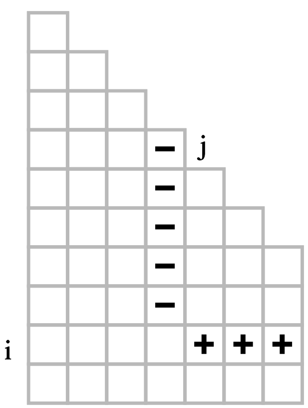
≥ 0

Trivial inequalities



$= 1$

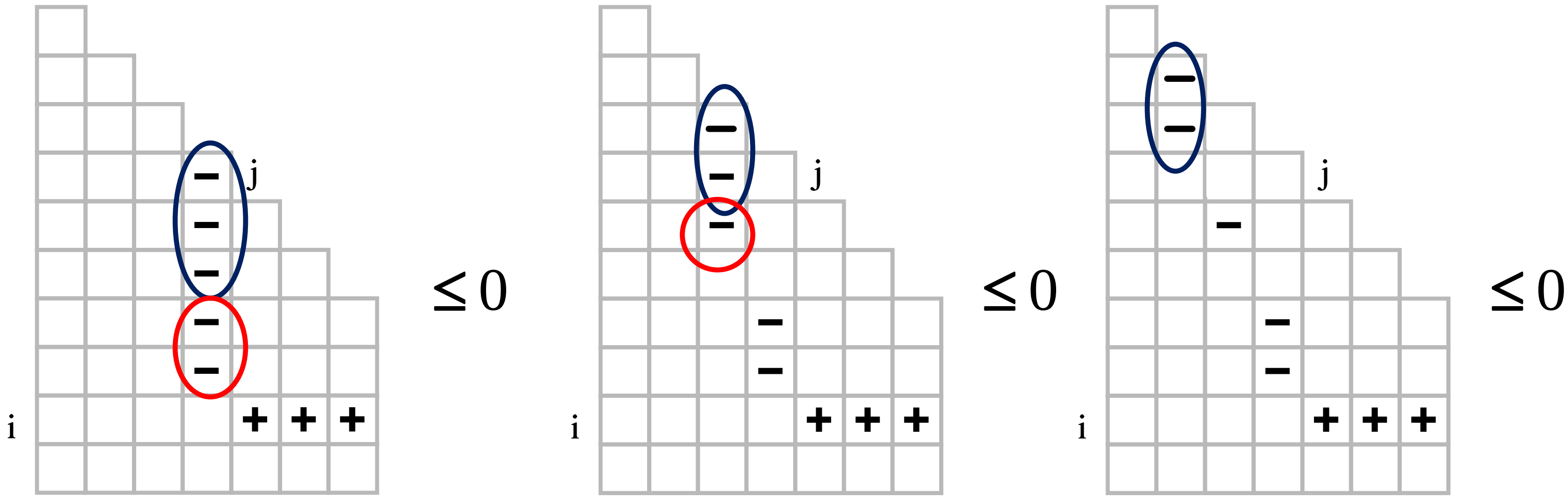
Row-sum equations

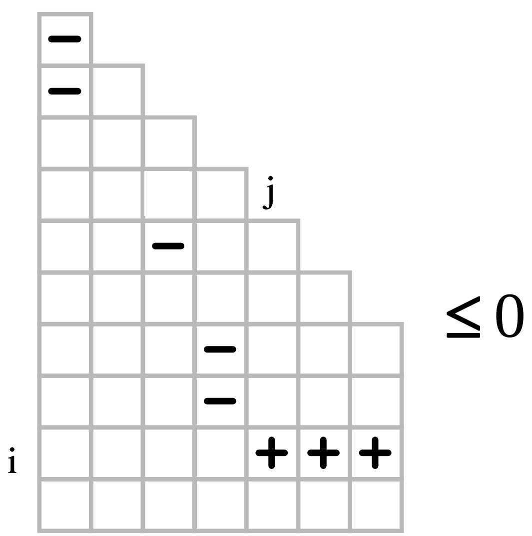


≤ 0

Column inequalities

Linear contraints for $O^{=}_{p,q}$ *(2)*





Shifted Column inequalities (SCI)
[Kaibel and Pfetsch, 06]

SCIs are the Chvátal-Gomory closure
of Column inequalities [Peinhardt, 09]

A complete linear description

SCI Thr. [Kaibel & Pfetsch, 06]
The Partitioning orbitope is completely described by:

- *Shifted-column inequalities*
- *Row-sum equations*
- *Trivial inequalities*

Moreover, it has an exponential number of facet-defining inequalities.

Comments:

- The LP has exponentially many constraints, but you can separate in time *O(pq)* [Kaibel & Pfetsch, 06]

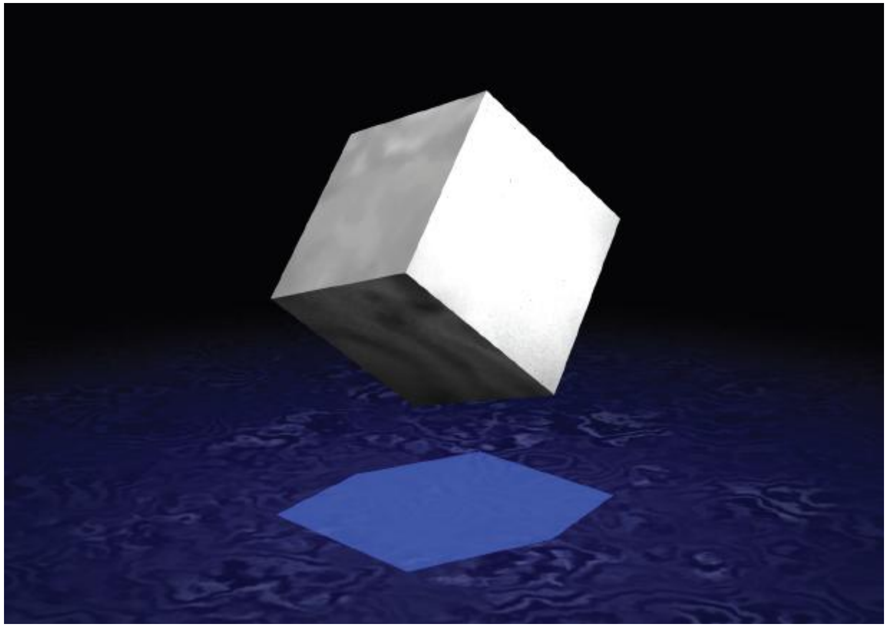
- Proof of the SCI Thr. is quite involved

Extended formulations

$$P = \{x \in \mathbb{R}^n | Ax \leq b\}$$

$$Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | Bx + Cy \leq d\}$$

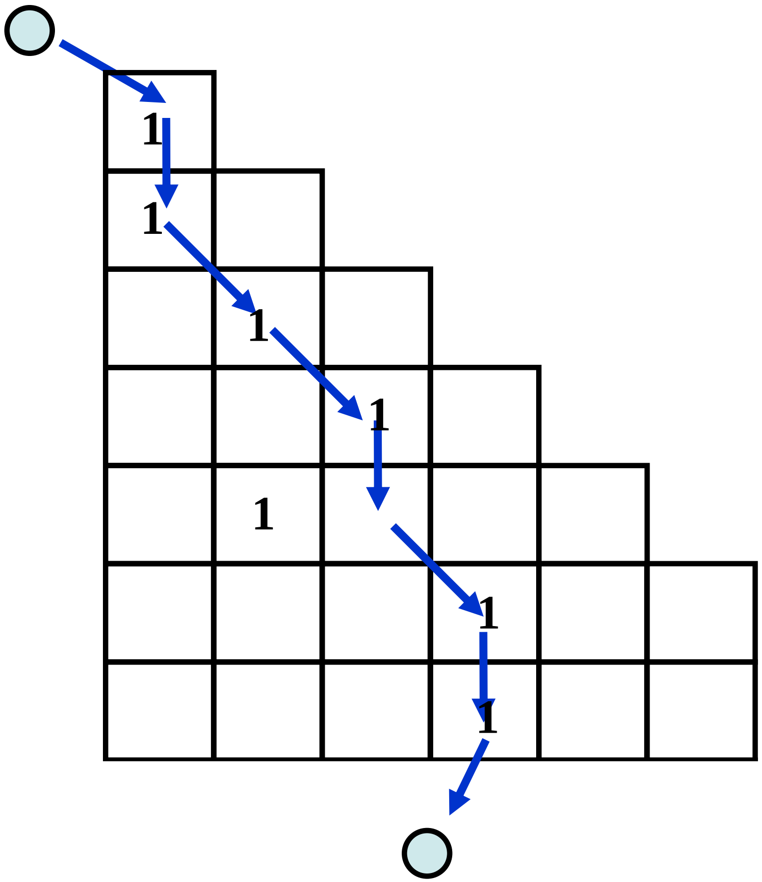
$$Proj_x(Q) = P$$

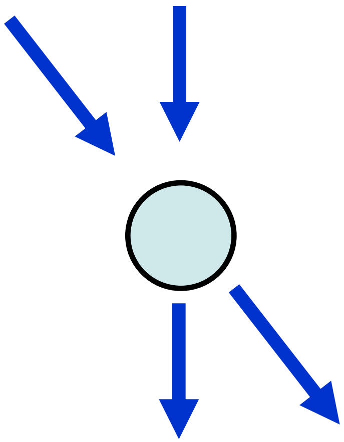


Lifting partitioning orbitopes (1)

Main idea:

Associate each integral $x \in \mathrm{O}^{=}_{p,q}$ with an s-t path on a digraph





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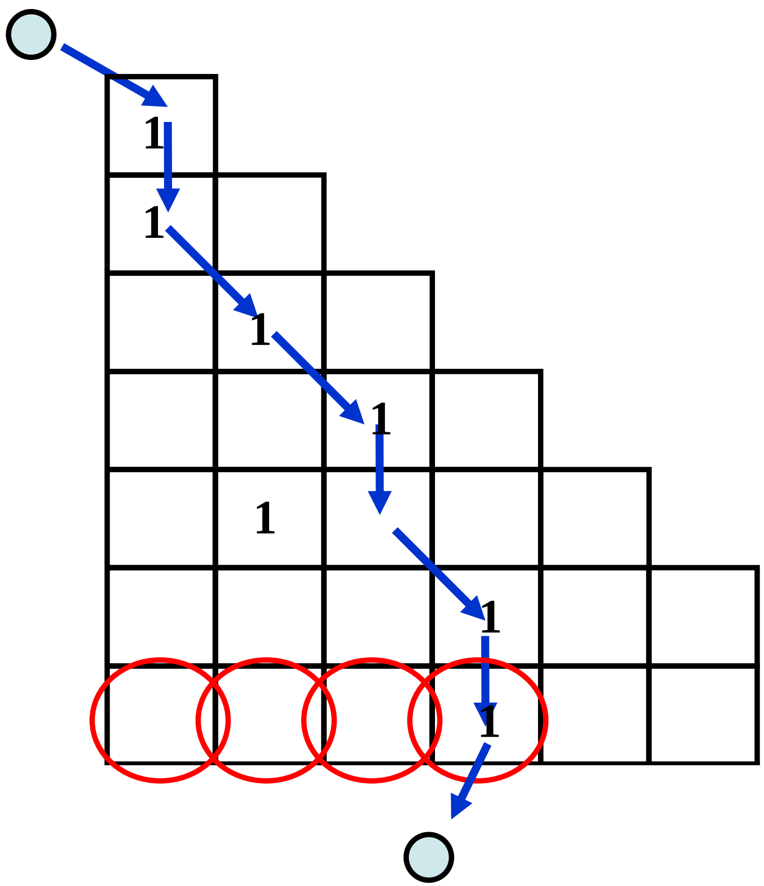
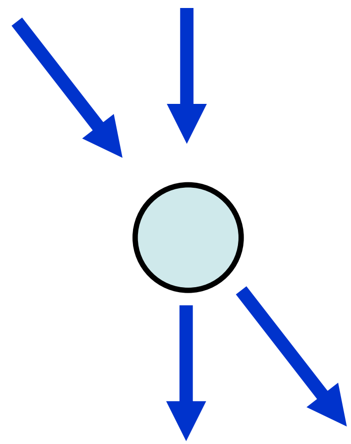
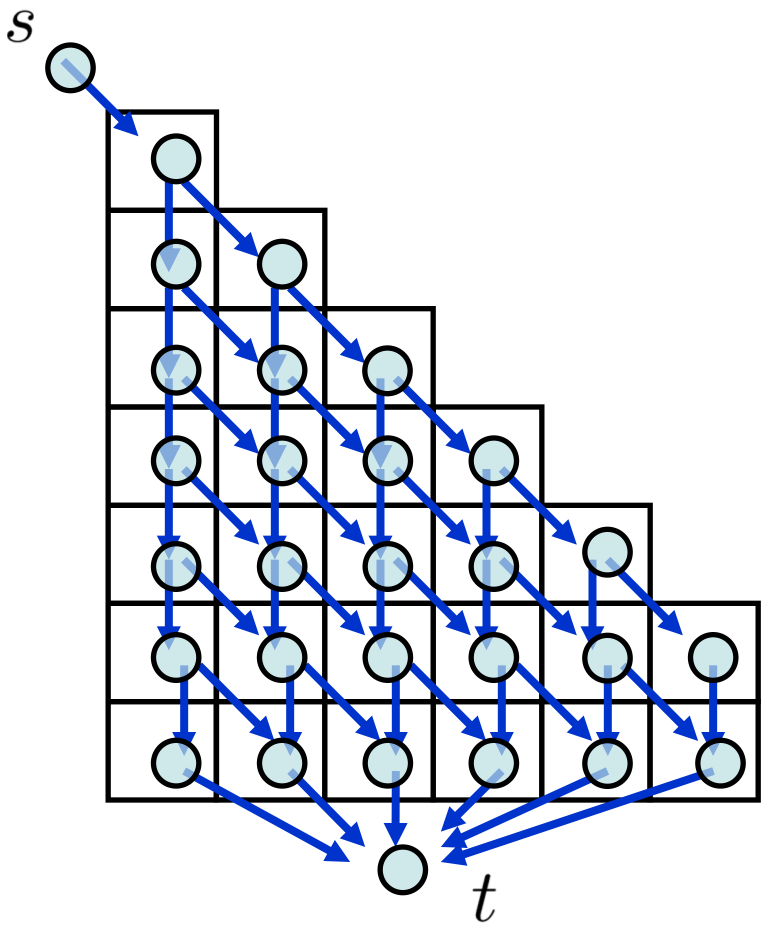


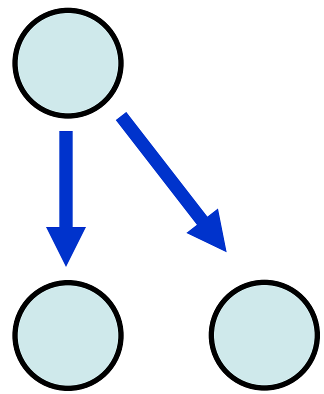


A single path corresponds to more vertices of $\mathrm{O}^{=}_{p,q}(G)$

Lifting partitioning orbitopes (2)



$$\max cx \text{ s.t. } x \in \mathrm{O}^{=}_{p,q}$$

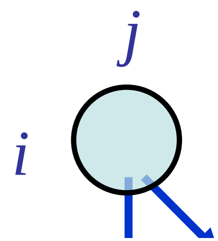


$$c^{\star}_{(i,j)\searrow} = c_{i+1,j+1}$$

$$c^{\star}_{(i,j)\downarrow} = \max_{\ell \leq j} c_{i+1,\ell}$$

Thr.(F. & Kaibel 08): The problem $\max cx$ s.t. $x \in \mathrm{O}^{=}_{p,q}$ can be solved in time $O(pq)$.

The extended formulation



$$y_{(i,j)^{\downarrow}}$$

$$y_{(i,j)^{\searrow}}$$

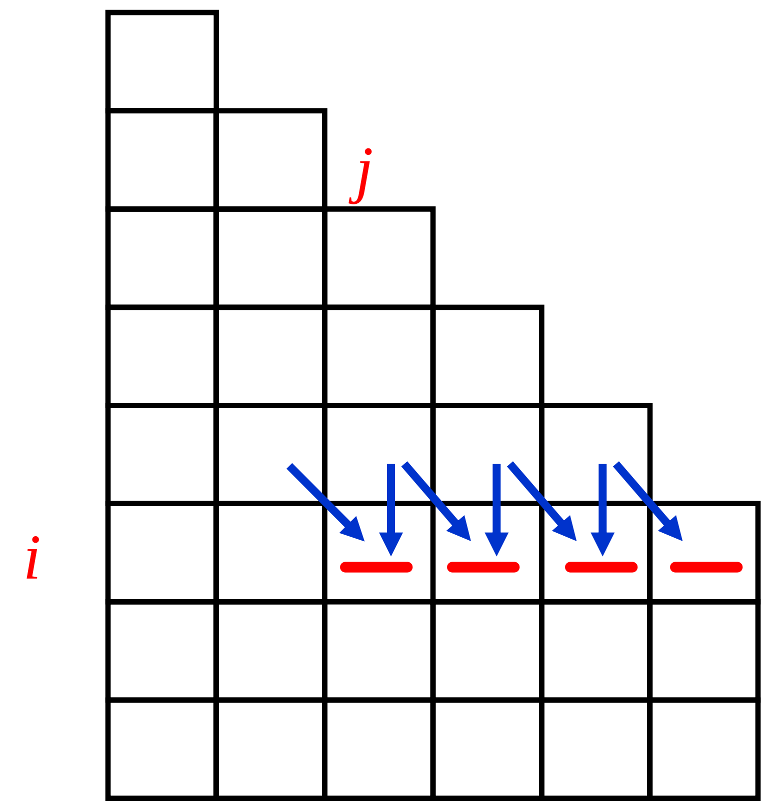
$$P_{p,q} = \{(x,y) \mid$$

$$y \in F_{p,q}$$

$$x(\mathrm{row}_i) = 1$$

$$y_{(i-1,j-1)^{\searrow}} \leq x_{i,j}$$

$$\sum_{m=j}^{i} x_{i,m} \leq y_{(i-1,j-1)^{\searrow}} + \sum_{m=j}^{i-1} (y_{(i-1,m)^{\searrow}} + y_{(i-1,m)^{\downarrow}}) \quad \}$$



$\lesseqgtr 0$

Thr.(F. & Kaibel 08): $P_{p,q}$ is an extended formulation for $\mathrm{O}^{=}_{p,q}$.

Integrality of $\mathcal{P}$ – sketch of the proof

Suppose you want to solve $\max cx$ s.t. $x \in O^{=}_{p,q}$

Define a new cost vector $c^\star$

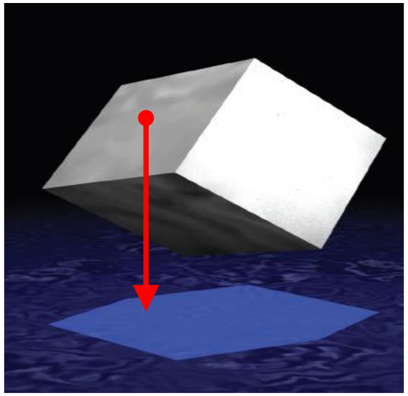
Claim1

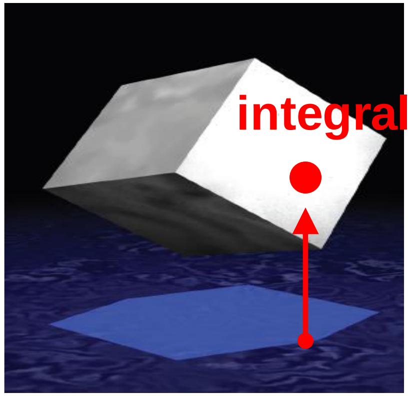
$$< c^\star, (0, y) > \;\geq\; < c, (x, y) > \quad \forall (x, y) \in P_{p,q}$$

Claim2

For each integral $y \in F_{p,q}$ exists integral x with

$(x, y) \in P_{p,q}$ and $< c, (x, y) > \; = \; < c^\star, (0, y) >$.

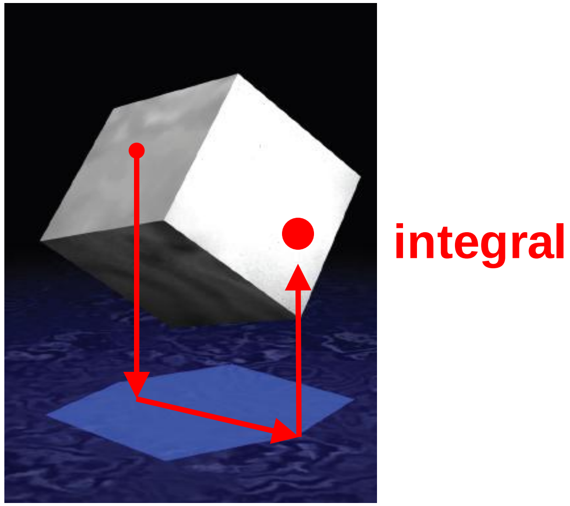




Integrality of $\mathcal{P}$ – sketch of the proof

Thus, for each $(\bar{x}, \bar{y}) \in P_{p,q}$:

$$
\begin{aligned}
< c, (\bar{x}, \bar{y}) > \quad &\leq \quad < c^\star, (0, \bar{y}) > \\
&= \quad < c^\star, (0, \tilde{y}) > \quad \text{for some integral } \tilde{y} \\
&= \quad < c, (\tilde{x}, \tilde{y}) > \quad \text{for } \tilde{x} \text{ integral}, (\tilde{x}, \tilde{y}) \in P_{p,q} \quad \square
\end{aligned}
$$



More on the Extended Formulation

- Once you prove integrality, projecting is easy
- after suitable transformations, we end up with a "very compact" formulation with less than $2pq$ variables, $4pq$ constraints and $10pq$ total nonzero elements.
- (almost) identical results hold for $\mathrm{O}^{\leq}_{p,q}$ (actually, all the work is done for $\mathrm{O}^{\leq}_{p,q}$)

Re-proving the SCI - theorem

The SCI-theorem is not necessary in our proofs

Use the extended formulation to obtain a new proof of the complete description in the original space. Why ?

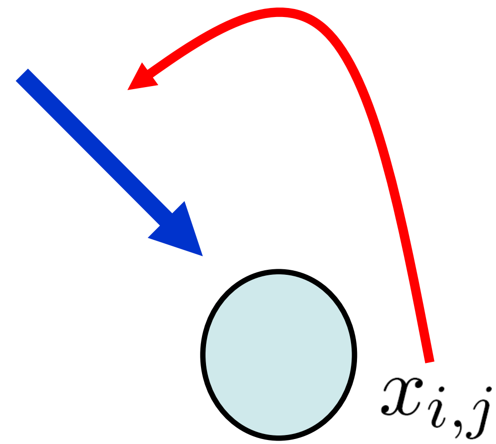
- Find a shorter proof
- Get new insight on the problem

Let $Q_{p,q}$ be the SCI-polytope. Since we already proved $\mathrm{Proj}_x(P_{p,q}) = \mathrm{O}^{=}_{p,q}$ we are left to prove

- $\mathrm{O}^{=}_{p,q} \subseteq Q_{p,q}$;
- $Q_{p,q} \subseteq \mathrm{Proj}_x(P_{p,q})$ ⇦

Re-proving the SCI-theorem (1)

SCI Thr. [Kaibel & Pfetsch, 06]
The Partitioning orbitope is completely described by:



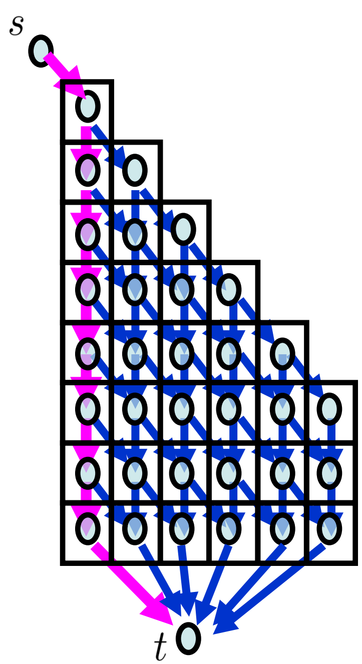
- *Shifted-column inequalities*
- *Row-sum equations*
- *Trivial inequalities*

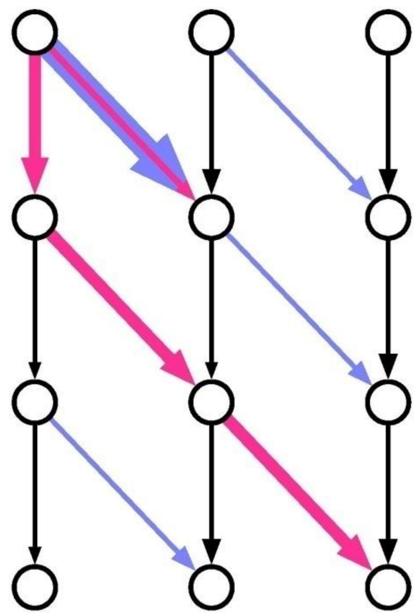
New proof: Given $x \in \mathrm{Q}_{p,q}$ we show that $(x, y = \Pi(x)) \in P_{p,q}$ for a suitable y

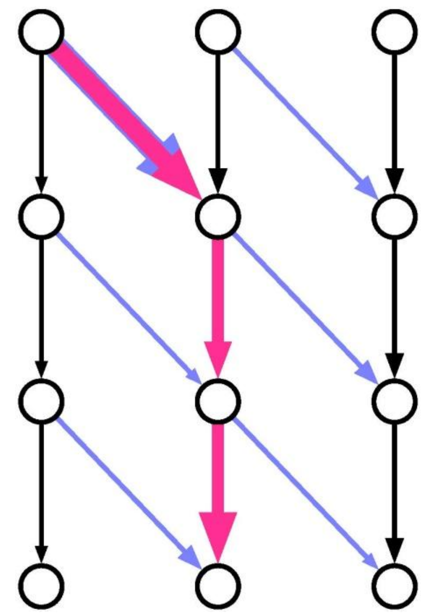
1) Let $x \in \mathrm{Q}_{p,q}$ and consider a network on digraph *D* with
   - capacity $+\infty$ on vertical arcs
   - capacity x_{ij} on the diagonal arc entering x_{ij}

Constructing the rightmost flow

2) Construct the *rightmost* flow $\Pi(x)$

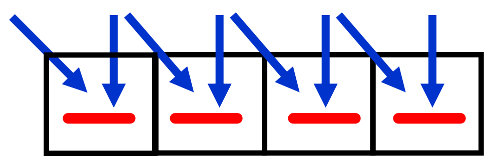




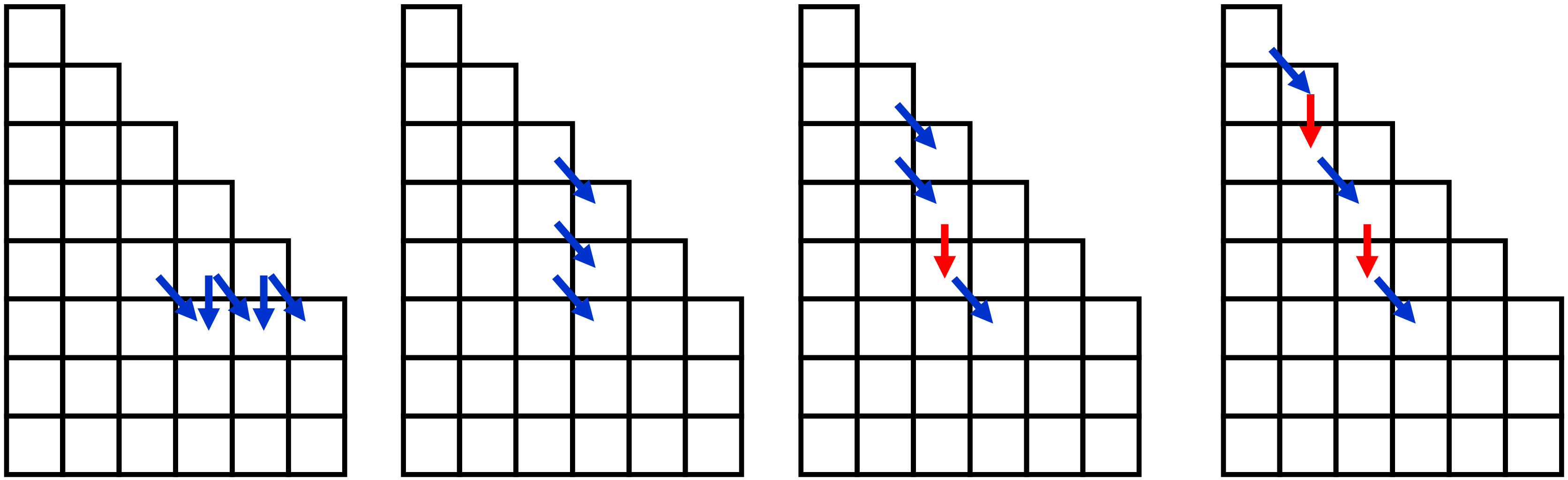


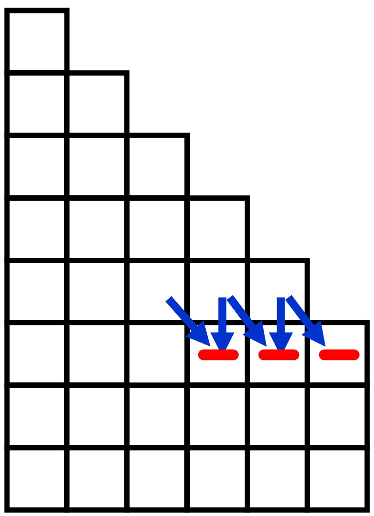
3) We shall prove that $(x, y = \Pi(x)) \in P_{p,q}$

4) Row-sum, $y_{(i-1,j-1)^{\searrow}} \leq x_{ij}$ come for free

5) We are left with  ≥ 0

Equivalent inequalities





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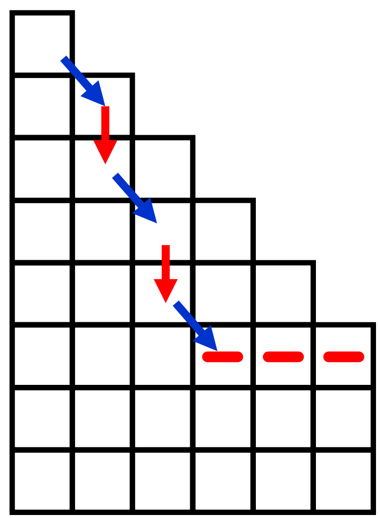
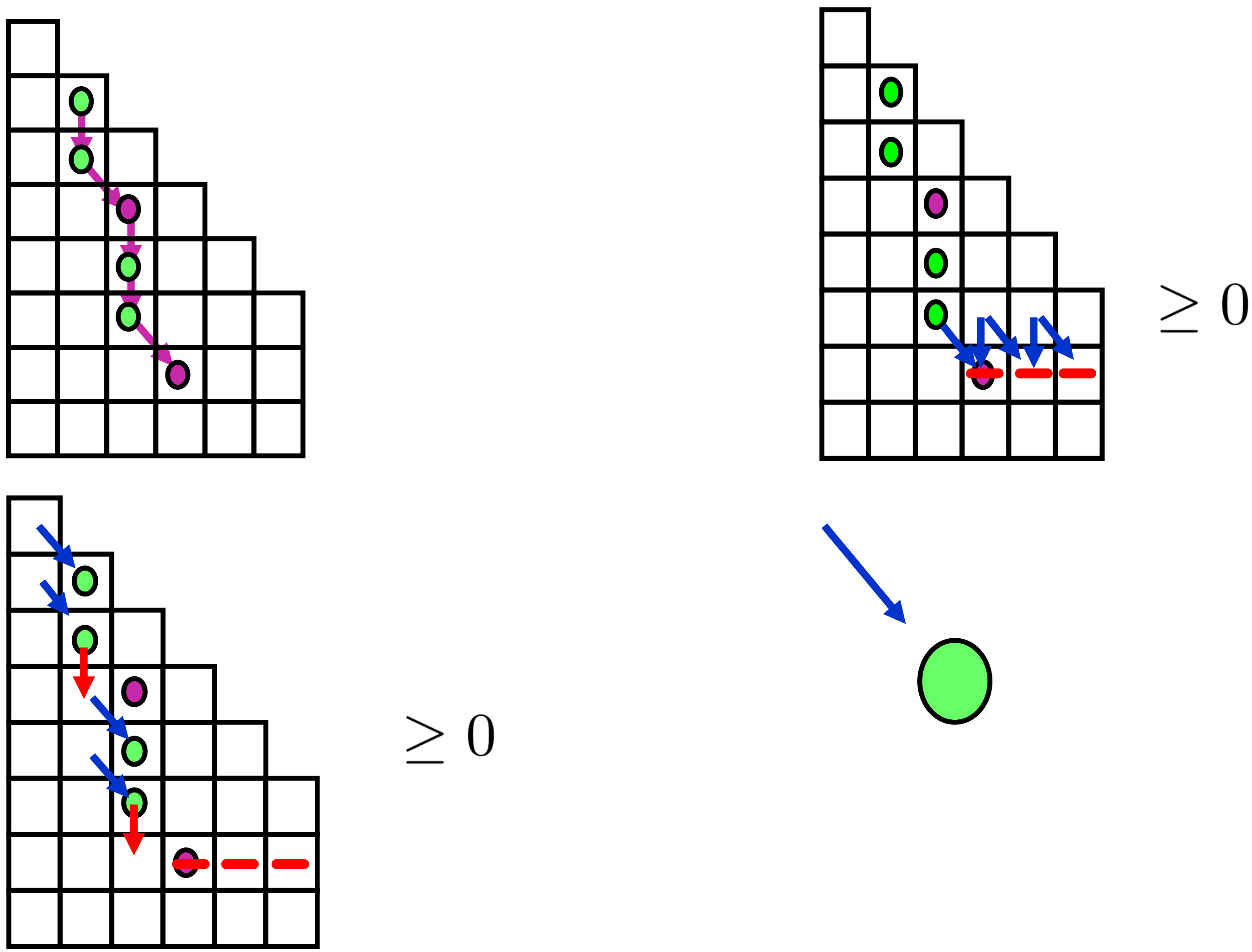


≥ 0

Re-proving the SCI-theorem (3)

6) Given *(i,j)*, build a backward *leftmost* flow



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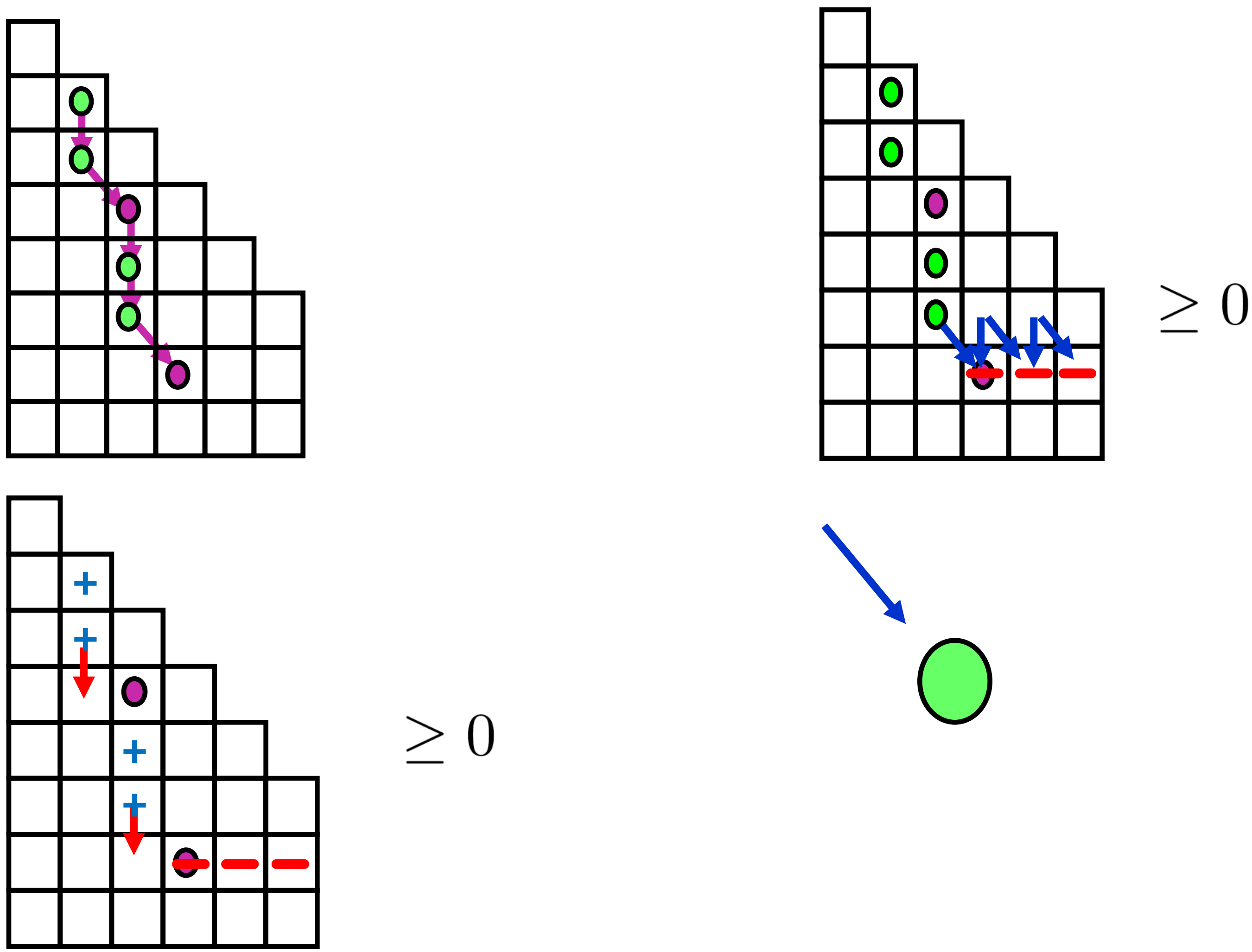

Re-proving the SCI-theorem (3)

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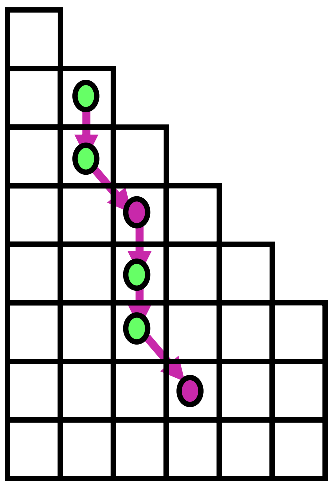

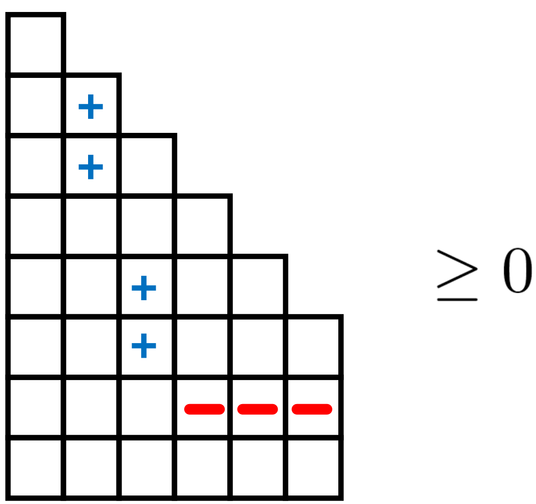

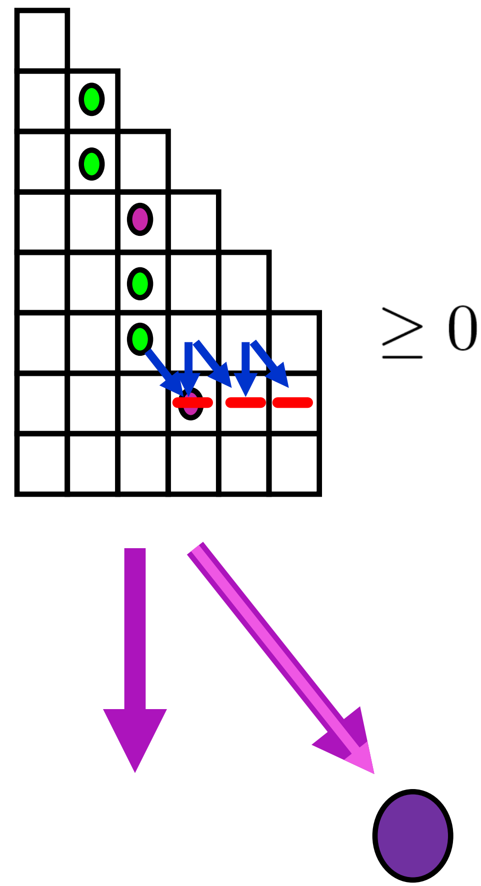

Shifted column inequality!

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Applications

- The “very compact” formulation has a linear number of variables and constraints. Can be used in practice ?

- An application-oriented result on orbitopes:

Orbitopal Fixing

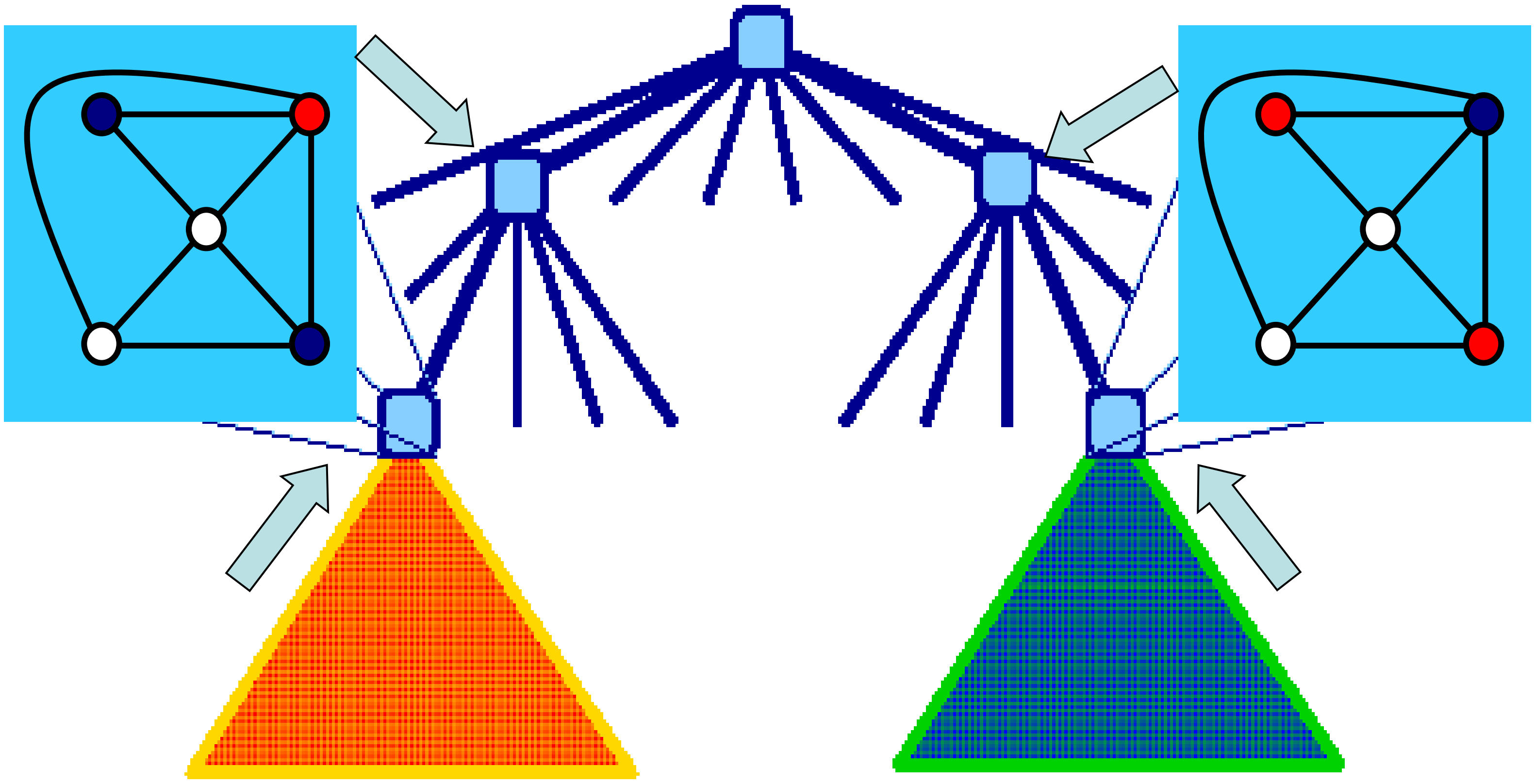
Orbitopal Fixing

1					
	0				
	0				
		0			
1	0	0	0	0	0
				0	
		0	0	0	
0	0	0	0		

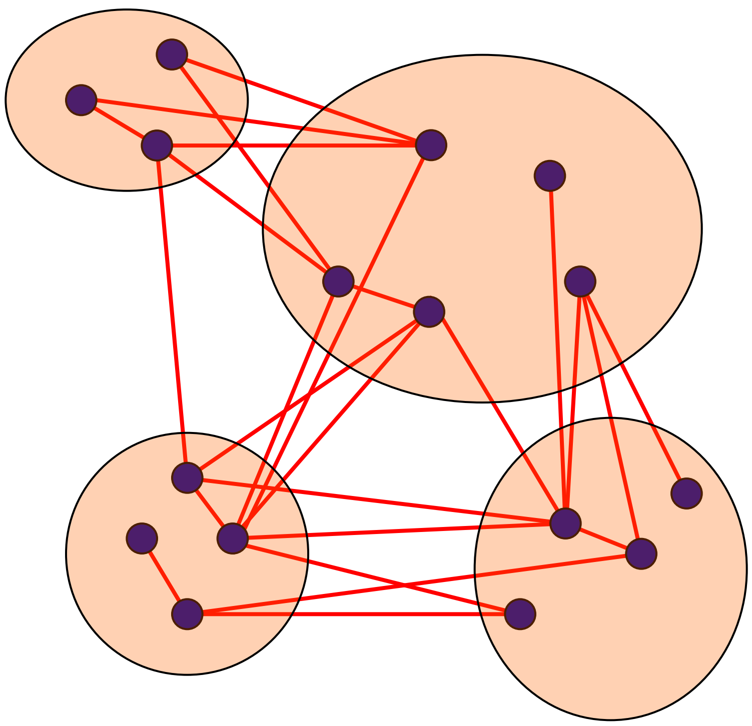
Thr. [Kaibel, Peinhardt, and Pfetsch, 07]**:**

Orbitopal fixing can be done in time $\mathrm{O}(pq)$

Orbitopal fixing in the branching tree



Application: the Graph partitioning problem

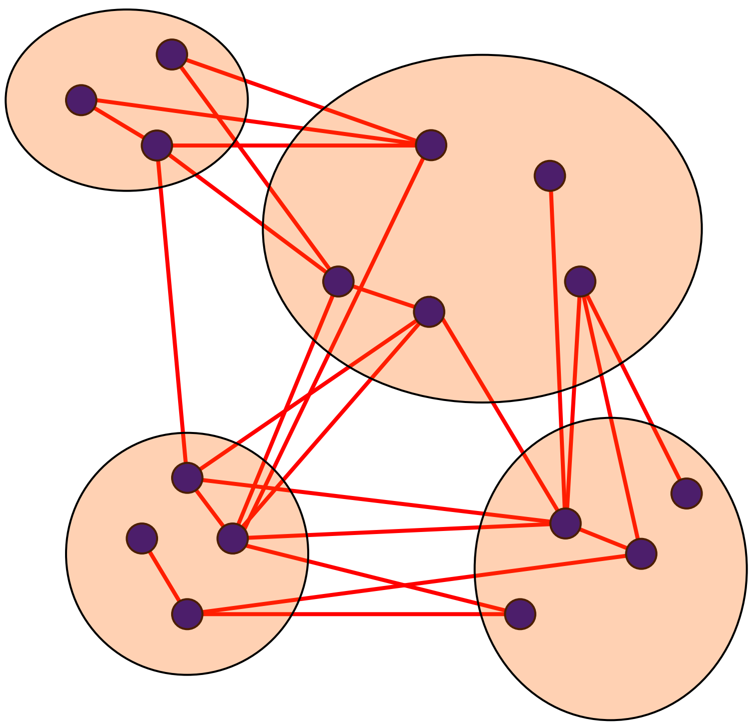


Input: a connected graph G (V,E) and nonnegative edge weights w on arcs, a strictly positive integer k.

k-partition: a partition of V in k subset

Graph partitioning problem: find the k-partition of V that minimizes the sum of the weights of the edges that connect nodes of the same subset.

Application: the Graph partitioning problem



$$\min \sum_{ik \in E} w_{ik} y_{ik}$$

subject to

$$\sum_{j=1}^{k} x_{ij} = 1$$

$$x_{ij} + x_{kj} - y_{ik} \leq 1 \quad \{i,k\} \in E$$

$$x, y \text{ binary}$$

Preliminary computational results (1)

graph	IsoP Time	OB Time	OF+SCI Time	Ext Time
2g_44_1601	0.0	0.0	0.0	0.0
2g_55_62	0.1	0.1	0.0	0.1
2g_66_66	0.1	0.1	0.1	0.1
2g_77_77	0.3	0.2	0.2	0.3
2g_88_88	0.6	0.5	**0.3**	0.3
2g_99_9211	1.3	1.0	**0.8**	1.7
2g_1010_824	1.0	1.0	0.9	1.3
2pm_44_44	0.0	0.0	0.0	0.0
2pm_55_55	0.0	0.0	0.1	0.1
2pm_66_66	0.1	0.1	0.1	0.1
2pm_77_777	0.1	0.1	0.1	0.1
2pm_88_888	0.3	**0.3**	0.5	0.4
2pm_99_999	1.6	**1.3**	1.7	1.5
3g_234_234	0.1	0.1	**0.1**	0.1
3g_333_333	0.1	0.1	0.1	0.1
3g_334_334	0.3	0.2	**0.2**	0.3
3g_344_344	0.6	0.5	0.4	0.5
3g_444_444	1.6	1.6	1.0	1.2
3pm_234_234	0.1	0.1	0.0	0.0
3pm_244_244	0.1	0.1	0.1	0.1
3pm_333_333	0.1	0.1	0.0	0.1
3pm_334_334	0.2	0.2	0.1	0.1
3pm_344_344	1.1	1.0	0.8	0.9
3pm_444_444	78.0	74.8	73.4	108.3
clique_20	0.1	0.1	0.1	0.1
clique_30	0.3	0.3	0.1	0.1
clique_40	0.4	0.4	0.5	0.5
clique_50	1.4	1.4	1.9	2.1
clique_60	**3.7**	3.7	4.3	4.7
clique_70	11.0	11.0	13.2	**9.8**

Implementations by Matthias Peinhardt,

Instances from

[Ferreira et al., 98]

Preliminary computational results (2)

graph	n	m	k	OF+SCI Time	Ext Time
cb450.47.8.99	47	99	47	11.9	7200.0
cb450.47.9.101	47	101	47	219.3	7200.0
cb450.61.9.187	61	187	61	383.6	7200.0
cb512.47.7.99	47	99	47	12.1	2584.8
cb512.47.8.101	47	101	47	31.6	7200.0

Implementations by Matthias Peinhardt,

Instances from [Ferreira et al., 98]

Outline

- Motivation: symmetric 0-1 formulations
- Partitioning Orbitopes in the original and in an extended space
- Applications
- More Orbitopes

More Orbitopes

- $O^{\leq}_{p,q}$: same results as those for $O^{=}_{p,q}$ **[Kaibel & Pfetsch, 06]** **[F. & Kaibel, 08]**

- $O_{p,q}$: exponential-size description in the original space for q=2, compact extended formulation **[Kaibel & Loos, 07]**

- $O^{\geq}_{p,q}$: NP-Hard **[Kaibel & Loos, 08]**

- $O^{\leq}_{p,\mathbf{q}}$ and $O^{=}_{p,\mathbf{q}}$: NP-Hard, there is an extended formulation of size $O(pq^r)$ where r is the nr. of groups **[F. & Kaibel, 08]**

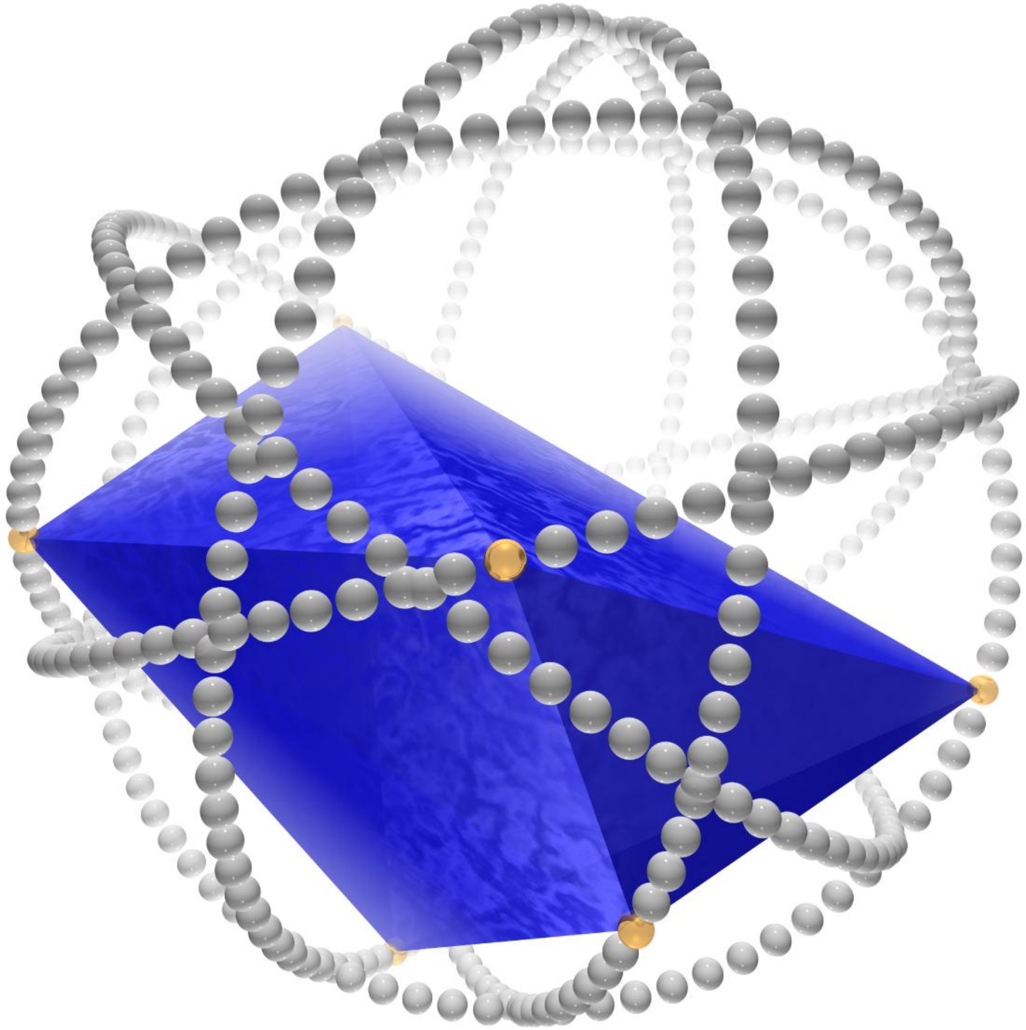
Change the group acting on the column, change constraints,…

Conclusions

Extended formulations for Packing and Partitioning Orbitopes…

- (very) compactly describe the polytopes;
- give more insight;
- shorten proofs;
- can be applied to solve real problems ?

Thank you for your attention



[Picture by Andreas Loos]