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Solution methods for the inventory routing problem

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We consider the so-called vendor-managed inventory (VMI) system in supply chain. :

- The supplier monitors the inventory and decides the replenishment policy of each retailer
- The supplier acts as a central decision maker who solves an integrated inventory-routing problem.

The advantage of a VMI policy with respect to the traditional retailer managed inventory policies lies in a more efficient resource utilization

- The supplier can reduce its inventories while maintaining the same level of service, or can increase the level of service and reduce the transportation cost through a more uniform utilization of the transportation capacities
- The retailers can devote fewer resources to monitoring their inventories and to placing orders, and have the guarantee that no stockout will occur

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We consider two replenishment strategies

- The order-up to level (OU):
  - Each retailer defines a minimum and a maximum inventory level and can be visited several times during the planning horizon.
  - the supplier monitors the inventory of each retailer and guarantees that no stockout will occur.
  - Every time a retailer is visited, the quantity delivered is such that the maximum inventory level is reached.

- The maximum level strategy (ML) :
  - Instead of imposing that every time a retailer is visited, the quantity delivered is such that the maximum level of inventor is reached, the only constraint on the shipping quantity is that it must be not greater than the maximum inventory level

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Inventory-routing is gaining in popularity, both from a practical standpoint and as a research area. This class of problems is rather difficult to solve.

Some surveys

Bertazzi, L., Speranza, M.G., Savelsbergh, M.W.P. (2008),
Inventory Routing,
in: Vehicle routing, Golden, B., Raghavan, R., Wasil, E. (eds.), to appear.

Campbell, A.M., Clarke, L., Kleywegt, A., Savelsbergh, M.W.P. (1998),
The Inventory Routing Problem,
in: Fleet Management and Logistics, Crainic, T.G., Laporte, G. (eds.), 95–113, Kluwer, Boston.

Federgruen, A., Simchi–Levi, D. (1995),
Analysis of Vehicle Routing and Inventory–Routing Problems,
in: Handbooks in Operations Research and Management Science, Ball, M.O., Magnanti, T.L., Monma, C.L., Nemhauser, G.L. (eds.), Vol. 8, 297–373, North–Holland.

We concentrate on a simplified version of the problem involving a single vehicle. This will hopefully serve as a basis for the understanding and resolution of more realistic cases.

Plan of the talk

Part I

An Branch-and-Cut algorithm

Archetti, Bertazzi, Laporte, Speranza

Transportation Science 41 (2007)

Part II

A Hybrid Heuristic

Archetti, Bertazzi, Hertz, Speranza

Submitted for publication in March 2009 (3 weeks ago)

(combination of a tabu search with the solution of MIP models)

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Part I

Problem description

We consider a logistic network in which a product is shipped from a common supplier 0 to a set M={1,2,…,n} of retailers over a time horizon H.

At each discrete time $t \in T=\{1,2,\ldots,H\}$

- a product quantity r_{0t} is produced or made available at the supplier
- a quantity r_{st} is consumed at retailer $s \in M$.

A starting inventory level B_0, at the supplier is given. $\boxed{B_t}$ is the inventory level at the supplier at period t.

Each retailer s defines a maximum inventory level U_s and has a given starting inventory level $I_{s0} \leq U_s$. $\boxed{I_{st}}$ is the inventory level at retailer s at period t.

If retailer s is visited at time t, then the quantity $\boxed{x_{st}}$ shipped to s depends on the replenishment policy

- OU policy : x_{st} is the difference between U_s and the current inventory level I_{st} of s
- ML policy : the quantity x_{st} can take any non-negative value that does not violate the capacity U_s

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Costs

The inventory cost is charged both at the supplier and at the retailers

- Denoting h_0 the unit inventory cost at the supplier, and by B_t the inventory level at the supplier at time t, the total inventory cost at the supplier is

$$\sum_{t \in T'} h_0 B_t$$

Where $T' = T \cup \{H+1\}$. The time H+1 is included in the computation in order to take into account the consequences of the operations performed at time H.

- Denoting h_s the unit inventory cost of retailer s, the total inventory cost at retailer s is

$$\sum_{t \in T'} h_s I_{st}$$

Shipments from the supplier to the retailers can be performed at any time $t \in T$ by a vehicle of capacity C. Each vehicle route visits visits all retailers that are served at the same time

- The transportation c_{ij} cost from i to j is known.
- Denoting $M' = M \cup \{0\}$ and $\boxed{y_{ij}^t}$ the binary variable equal to 1 if j immediately follows i in the route traveled at time t, and 0 otherwise, the total transportation cost is

$$\sum_{\substack{i,j \in M' \\ j<i}} \sum_{t \in T} c_{ij} y_{ij}^t$$

Objective function to be minimized

$$\sum_{t\in T'} h_0 B_t + \sum_{t\in T'}\sum_{s\in M} h_s I_{st} + \sum_{\substack{i,j\in M'\\ j<i}} \sum_{t\in T} c_{ij} y_{ij}^t$$

Constraints

1. Inventory definition at the supplier

$$B_t = B_{t-1} + r_{0t-1} - \sum_{s\in M} x_{st-1} \qquad t\in T'$$

(where $x_{s0}=r_{00}=0$)

2. Stockout constraint at the supplier

$$B_t \geq \sum_{s\in M} x_{st} \qquad t\in T$$

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3. Inventory definition at the retailers

$$I_{st} = I_{st-1} + x_{st-1} - r_{st-1} \qquad s \in M \quad t \in T'$$

(where $r_{s0}=0$)

4. Stockout constraint at the retailers

$$I_{st} \geq 0 \qquad s \in M \quad t \in T'$$

5. Order-up-to level constraints

$$x_{st} \geq U_s z_{st} - I_{st} \qquad s \in M \quad t \in T$$

$$x_{st} \leq U_s - I_{st} \qquad s \in M \quad t \in T$$

$$x_{st} \leq U_s z_{st} \qquad s \in M \quad t \in T$$

$\boxed{z_{st}}$ is a binary variable that equals 1 if retailer s is served at time t, and 0 otherwise.
$\boxed{z_{0t}}$ is a binary variable that equals I if the supplier delivers something at time t.

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6. Capacity constraints

$$\sum_{s \in M} x_{st} \leq C \qquad t \in T$$

7. Routing constraints

- if at least one retailer is visited at time t, then the route traveled at time t has to "visit" the supplier. z_{0t} equals 1 in such a case, and 0 otherwise

$$\sum_{s \in M} x_{st} \leq C z_{0t} \qquad t \in T$$

- if deliveries are made at time t, then the route traveled at time t has to contain one arc entering every vertex i on the route and one arc leaving every i.

$$\sum_{\substack{j \in M' \\ j<i}} y_{ij}^{t} + \sum_{\substack{j \in M' \\ j>i}} y_{ji}^{t} = 2 z_{it} \qquad i \in M' \quad t \in T$$

- Subtours elimination constraints

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8. Nonnegativity and integrality constraints

$$x_{st} \geq 0 \qquad i \in M \quad t \in T$$

$$y_{ij}^{t} \in \{0,1\} \qquad i,j \in M \; j<i \quad t \in T$$

$$y_{i0}^{t} \in \{0,1,2\} \qquad i \in M \quad t \in T$$

$$z_{it} \in \{0,1\} \qquad i \in M' \quad t \in T$$

Additional valid inequalities

9. If s is not served in the times t-k,t-k+1,…,t, then the inventory level $I_{st\text{-}k}$ at s at time t-k is $\geq \sum_{j=0}^{k} r_{st-j}$

$$I_{st-k} \geq \left(1 - \sum_{j=0}^{k} z_{st-j}\right) \sum_{j=0}^{k} r_{st-j} \qquad s \in M \quad t \in T \quad k=0,\ldots,t-1$$

10. This constraint is valid only for the OU policy. If t-k is the last time retailer s was visited before time t, then $I_{st}=U_s-\sum_{j=t-k}^{t-1} r_{sj}$

$$I_{st} \geq U_s z_{st-k} - \sum_{j=t-k}^{t-1} r_{sj} \qquad s\in M \quad t\in T \quad k=0,\ldots,t-1$$

11. If retailer s is is visited at time t, then the supplier has to be included in the route traveled at time t

$$z_{st} \leq z_{0t} \qquad s\in M \quad t\in T$$

12. If the supplier is the successor of retailer i in the route traveled at time t (i.e., $y_{i0}^{t}=1$ or 2), then i has to be visited at time t.

    If retailer j is the successor of retailer i in the route traveled at time t (i.e., $y_{ij}^{t}=1$), then i has to be visited at time t.

$$y_{i0}^{t} \leq 2z_{it} \qquad i\in M \quad t\in T$$

$$y_{ij}^{t} \leq z_{it} \qquad i,j\in M \quad t\in T$$

An exact algorithm

Archetti, Bertazzi, Laporte and Speranza (Transportation Science 41 (2007) have developed a Branch-and-Cut algorithm to solve the problem to optimality

- they consider all constraints and inequalities, except the subtour elimination constraints
- All valid inequalities are included in the root node.
- Branching occurs in priority on variables z_{jt} and then on variables y_{ij}^{t}.
- The search is developed according to a best bound first strategy.
- An initial upper bound is obtained using a heuristic developed by **Bertazzi, Paletta and Speranza** (Transportation Science 36 (2002)

Short description of the heuristic of Bertazzi et al.

1. the retailers are ranked in non-decreasing order of the average number of time units needed to consume the quantity U_s (and in non-increasing order of U_s in case of equality)

2. In the initialization phase, a feasible solution is constructed by means of an iterative procedure that inserts a retailer at each iteration
   - When retailer s is considered, a set of delivery times is determined by a solving a shortest-path problem on an acyclic network in which every vertex is a possible delivery time
   - For each of the selected delivery times, the retailer is inserted in the route traveled that day by applying the cheapest insertion criterion

3. In the second phase, the solution is improved iteratively
   - At each iteration, a pair of retailers is removed and reinserted. If this reduces the total cost, the solution is modified accordingly.

Part II

General description of the hybrid heuristic

Apply the Initialization procedure to generate an initial solution s and set $s_{best} \leftarrow s$.

While the number of iterations without improvement of s_{best} is $\leq$ MaxIter **do**

- Apply the Move procedure to find the best solution s' in the neighbourhood N(s) of s.
- **If** s' is better than s_{best} **then**

  Apply the Improvement procedure to possibly improve s' and set $s_{best} \leftarrow s'$.
- Set $s \leftarrow s'$
- **If** the number of iterations without improvement of s_{best} is a multiple of JumpIter **then**

  Apply the Jump procedure to modify the current solution s

End while.

There are two parameters

- MaxIter to indicate when to stop
- JumpIter to indicate when to jump to a new region of the search space

There are four basic procedures

- Initialization : it generates a starting solution
- Move : it generates the best neighbour s’ of s in N(s)
- Improvement : it tries to improve s_{best} by solving MIP problems
- Jump : it performs changes on s in order to jump to a new regions of the search space

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Definition of the search space

A solution to the inventory routing problem is $\boxed{\text{feasible}}$ is

- there is no stockout at the supplier,
- there is no stockout at the retailers,
- the level of the inventory of each retailer is never greater than its maximum level
- there is no violation of the vehicle capacity constraint

The search space visited by the heuristic contains solutions which are not necessarily feasible.

- there is no stockout at the retailers
- the level of the inventory of each retailer is never greater than its maximum level
- stockout at the supplier is permitted
- the vehicle capacity is possibly exceeded at some time periods

Such solutions are called $\boxed{\textbf{admissible}}$.

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The objective function to be minimized is the sum of

- the inventory and transportation costs,
- + two penalty terms related to infeasibility.

Penalty for a stockout a the supplier $$\beta \sum_{t \in T'} \max\{0, -B_t(s)\}$$

Penalty for the violation of the vehicle capacity constraint $$\alpha \sum_{t \in T} \max\{0, Q_t(s) - C\}$$

where, for a solution s,

- $B_t(s)$ the inventory level at the supplier at time t
- $Q_t(s)$ the total quantity delivered at time t

Procedure Initialization

Each retailer is considered sequentially and the delivery times are set as late as possible, before a stockout situation occurs.

Such a solution is obviously **admissible**, but not necessarily feasible.

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Procedure Move

A neighbour s' of s is obtained by adding and/or removing visits at some retailers. These changes are performed as follows

Removal of a visit

When we remove a visit to retailer i at time t, we first remove retailer i from the vehicle route at time t and its predecessor is linked to its successor.

- In the case of the OU policy, the quantity delivered to i at time t is transferred to the following visit (if any). Such a removal is performed only if it creates no stockout at customer i in order to keep the solution admissible

- For the ML policy,

  if there is no stockout at i when removing the visit, **then** nothing else is made;

  otherwise, the removal is performed only if the stockout at i can be avoided by increasing the quantity delivered at the previous visit (if any) to a value not larger than the maximum capacity U_i.

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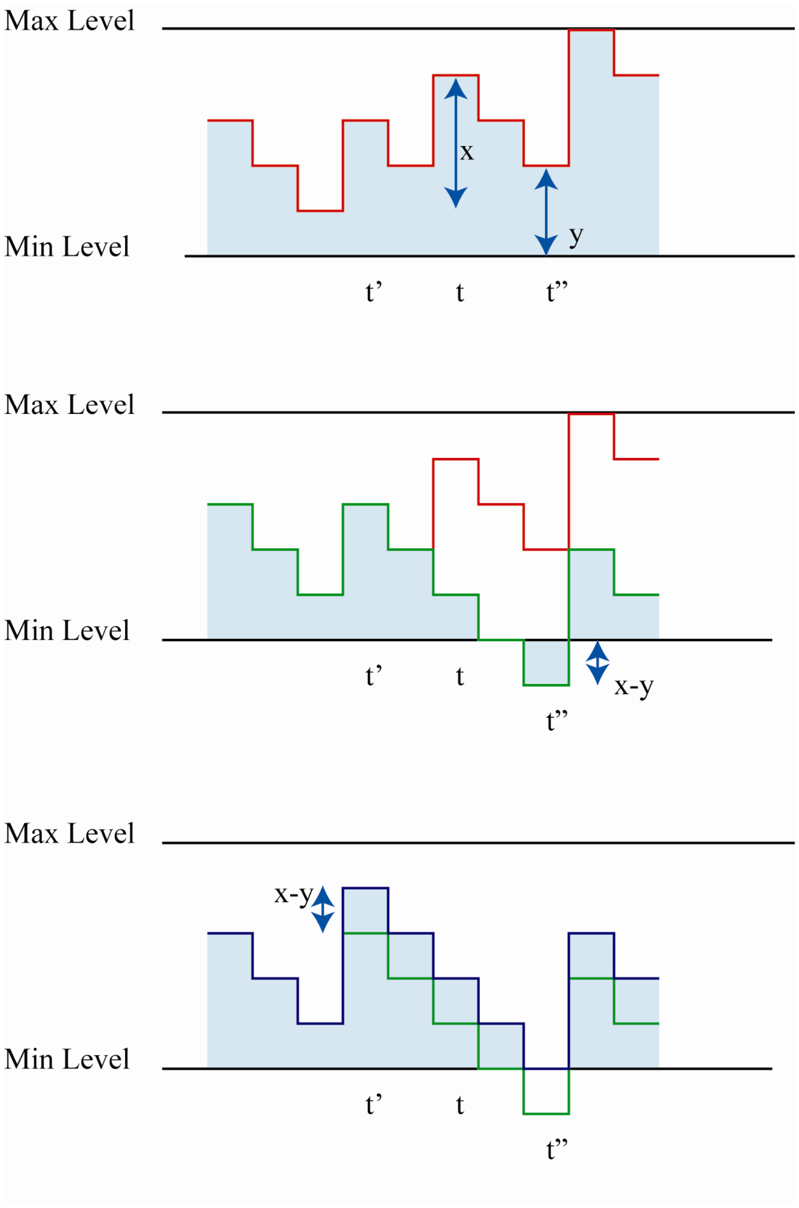

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Insertion of a visit

When we insert a visit to retailer i at time t, we first add retailer i to the vehicle route at time t using the cheapest insertion method.

- In the case of the OU policy, the quantity delivered to i at time t is equal to the difference between U_i and $I_{it}(s)$. The same quantity is removed from the next visit (if any)
- For the ML policy, we compute the minimum value between
  - $U_i - I_{it}(s)$
  - The residual capacity of the vehicle at time t
  - The quantity available at the supplier at time t

  If this minimum is zero, then r_{it} units (the demand at time t) are delivered.

  In all cases, the case quantity is removed from the next visit (if any).

We consider two tabu lists L_a and L_r

- If retailer i is visited at time t in s but not in s' then the pair (i,t) is introduced in L_a and it is forbidden for some iterations to add delivery time t at retailer i
- If retailer i is visited at time t in s' but not in s, then the pair (i,t) is introduced in L_r and it is forbidden for some iterations to remove delivery time t at retailer i

Construction of the neighbourhood N(s)

This is done in two steps

Step 1 : Construction of a set N'(s) of admissible solutions

N'(s) contains all solutions which can be obtained from s by one of the following simple changes:

- Removal of a visit at a retailer
- Insertion of a visit at a retailer
- Move of a visit at a retailer i from time t to time t' (so that i is not yet visited at time t')
- Swap of a visit to retailer i a time t with a visit to retailer j at time t' (so that i is not visited at time t' and j is not visited at time t)

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Step 2 : Construction of N(s) by improving the solutions in N'(s).

Consider any solution s' in N'(s).

- If $h_i > h_0$ it might be interesting to remove visits to i since this will strictly decrease the total inventory cost and the transportation cost without creating any stockout at the supplier.
- Suppose there is a visit to i at time t. If $Q_t(s') > C$ or $B_t(s') < 0$, then the removal of a visit to i at time t reduces the penalty component of the objective function.

If such a change induces an admissible solution with a strict decrease of the objective function, then the move is performed:

For the ML policy, we also test if $h_i > h_0$. In such a case it might be interesting to increase the quantity delivered to i at time t. We increase the delivery as much as possible without exceeding the maximum inventory level if such a change gives a solution with a strict decrease of the objective function.

Procedure Improvement

We solve two MIPs

1. Given a solution s, we try to improve it by assigning routes to different time periods, without adding new retailers in the routes.
2. Given a solution s, we do not change the time assigned to each vehicle route, whereas the removal of insertion of customers into routes is allowed.

First MIP

The only possible changes are the removal of customers from a route and the assignment of routes to different times. We denote R the set of routes in s.

Data

Δ_{ir} = transportation saving if i is removed from route r. (join the predecessor with the successor)

σ_{ir} =1 if i belongs to route r, 0 otherwise.

Variables

- w_{ir} = 1 if i is removed from route r, 0 otherwise
- z_{it} = 1 if i is visited at time t, 0 otherwise
- d_{rt} =1 if route r is assigned to time t, 0 otherwise

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Objective :

Minimize $$\sum_{t\in T'} h_0 B_t + \sum_{t\in T'}\sum_{i\in M} h_i I_{it} - \sum_{i\in M}\sum_{r\in R} \Delta_{ir} w_{ir}$$

Constraints

$$B_t = B_{t-1} + r_{0t-1} - \sum_{s\in M} x_{st-1} \qquad t\in T'$$

$$B_t \geq \sum_{s\in M} x_{st} \qquad t\in T$$

$$I_{st} = I_{st-1} + x_{st-1} - r_{st-1} \qquad s\in M \quad t\in T'$$

$$x_{st} \geq U_s z_{st} - I_{st} \qquad s\in M \quad t\in T \quad \text{(OU)}$$

$$x_{st} \leq U_s - I_{st} \qquad s\in M \quad t\in T$$

$$x_{st} \leq U_s z_{st} \qquad s\in M \quad t\in T \quad \text{(OU)}$$

$$\sum_{s\in M} x_{st} \leq C \qquad t\in T$$

+ Nonnegativity and integrality constraints

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+ A route can be assigned to at most one time period

$$\sum_{t \in T} d_{rt} \le 1 \qquad r \in R$$

There is at most one route at each time t

$$\sum_{r \in R} d_{rt} \le 1 \qquad t \in T$$

A retailer can be visited at time t only if it is visited by the route assigned at time t

$$x_{it} \le U_i \sum_{r \in R} \sigma_{ir} d_{rt} \qquad i \in M \quad t \in T$$

A retailer cannot be visited at time t if it is currently visited at time t but is removed from this route.

$$x_{it} \le U_i (2 - \sigma_{ir}(d_{rt} + w_{ir})) \qquad i \in M \quad t \in T \quad r \in R$$

A retailer can be removed only from a route where it is visited.

$$w_{ir} \le \sigma_{ir} \sum_{t \in T} d_{rt} \qquad i \in M \quad r \in R$$

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Second MIP

The only possible changes are the removal and the insertion of retailers into routes, whereas we do not change the time of each vehicle route.

Data

Δ_{it} = transportation saving if i is removed from the route at time t.

Γ_{it} = insertion cost of retailer i into the route at time t (cheapest insertion).

σ_{it} =1 if i is visited at time t in s, 0 otherwise.

Variables

- w_{it} = 1 if i is removed from the route at time t, 0 otherwise
- v_{it} = 1 if i is inserted into the route at time t, 0 otherwise
- z_{it} = 1 if i is visited at time t, 0 otherwise

Objective :

Minimize
$$\sum_{t\in T'} h_0 B_t + \sum_{t\in T'}\sum_{i\in M} h_i I_{it} - \sum_{i\in M}\sum_{t\in T} \Delta_{it} w_{it} + \sum_{i\in M}\sum_{t\in T} \Gamma_{it} v_{it}$$

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Constraints

$$B_t = B_{t-1} + r_{0t-1} - \sum_{s\in M} x_{st-1} \qquad t\in T'$$

$$B_t \geq \sum_{s\in M} x_{st} \qquad t\in T$$

$$I_{st} = I_{st-1} + x_{st-1} - r_{st-1} \qquad s\in M \quad t\in T'$$

$$x_{st} \geq U_s z_{st} - I_{st} \qquad s\in M \quad t\in T \quad \text{(OU)}$$

$$x_{st} \leq U_s - I_{st} \qquad s\in M \quad t\in T$$

$$x_{st} \leq U_s z_{st} \qquad s\in M \quad t\in T \quad \text{(OU)}$$

$$\sum_{s\in M} x_{st} \leq C \qquad t\in T$$

\+ Nonnegativity and integrality constraints

A retailer cannot be inserted into a route that already visits him

$$v_{it} \leq 1 - \sigma_{it} \qquad i\in M \quad t\in T$$

A retailer cannot be removed from a route that does not visit him

$$w_{it} \leq \sigma_{it} \qquad t\in T$$

A retailer can be visited at time t only if it is visited by the route assigned at time t and is not removed from it, or if he is inserted into the route at time t.

$$x_{it} \leq U_i(\sigma_{it} - w_{it} + v_{it}) \qquad i\in M \quad t\in T$$

Remark

If more than one retailer is removed from a route or inserted (for the second MIP), then the variation of the transportation cost in the objective function is only an estimation of the real gain or loss induced by such a modification.

Theorem These two MIPs correspond to NP-hard problems

Proof

Knapsack problem $\propto$ First MIP

Partition problem $\propto$ Second MIP

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Procedure Jump

Retailers are moved from time periods where they are typically visited to time periods where they are typically not visited.

More precisely, as long as there exists a triplet (i,t,t') such that

- i is a retailer visited at time t since at least JumpIter / 2 iterations
- i was never visited at time t' ≠ t during the last JumpIter / 2 iterations
- the move of the visit to i from time t to time t' does not create a stockout at i

we take such a triplet at random and perform the move.

When no more changes of this kind can be performed, we apply the second MIP and consider the resulting solution as the new current solution.

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Computational experiments

Small instances

- n=5 to 50 customers
- H=3 and 6

160 instances

Comparison of the exact algorithm with the hybrid heuristic

- Average error of our hybrid heuristic

  OU policy : 0.08%

  ML policy : 0.05%

  Maximal error of our hybrid heuristic

  OU policy : 1.84%

  ML policy : 1.17%

- The heuristic of Bertazzi, Paletta and Speranza (OU policy)

  Average error : 2.86%

  Maximal error : 14.52%

- For H=3 with the ML policy, only one instance is not solved to optimality by the hybrid heuristic

- For H=3, the hybrid heuristic finds 88% optimal solutions

  For H=6, the success rate is 52.5%

Bigger instances

- Up to 200 retailers
- H=3 and 6

60 instances

OU policy

- The heuristic of Bertazzi, Paletta and Speranza always takes less than 3 minutes
- We have run our algorithms for 5 minutes, 10 minutes, 30 minutes and 1hour

  The solution produced by each algorithm is compared to the best solution obtained during our experiments

	Average deviation	Worst deviation
BPS	2.74%	10.3%
Hybrid 5 min	1.64%	7.44%
Hybrid 10 min	1.33%	6.07%
Hybrid 30 min	0.72%	5.82%
Hybrid 1 hour	0.07%	2.37%

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ML policy

	Average deviation	Worst deviation
Hybrid 5 min	0.34%	2,35%
Hybrid 10 min	0.25%	2.35%
Hybrid 30 min	0.04%	0.78%
Hybrid 1 hour	0.00%	0.00%

Québec

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