A Survey on Results for the Stable Set Polytope of Claw-Free Graphs

Annegret K. Wagler

Institute for Mathematical Optimization (IMO)
Otto-von-Guericke-Universität Magdeburg
Germany

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Outline

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The stable set problem

Stable set S

set of pairwise non-adjacent nodes of a graph G

Stable set problem

determine a stable set of maximum cardinality or weight in a graph G

Problem (Grötschel, Lovász & Schrijver 1988)

Consider the **stable set polytope**

$$\text{STAB}(G) = \text{conv}\{\chi^S \in \{0,1\}^{|G|} : S \subseteq G \text{ stable set}\}$$

and find a representation

$$\text{STAB}(G) = \{x \in \mathbf{R}_+^{|G|} : Ax \leq b\}$$

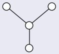
via a **facet-defining system** in order to compute the stability number

$$\alpha(G,c) = \max c^T x, x \in \text{STAB}(G)$$

as a **linear program**.

The stable set problem for claw-free graphs

Definition

A graph G is claw-free if G does not contain  as induced subgraph.

The stable set problem for claw-free graphs is "asymmetric"
as it

can be solved in polynomial time by combinatorial algorithms of

- Minty (1980)
- Sbihi (1980)
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- there can occur arbitrarily complicated facets and
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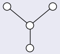
A graph G is claw-free if G does not contain  as induced subgraph.

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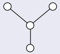
A graph G is claw-free if G does not contain  as induced subgraph.

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Clique constraints and perfect graphs

Clique constraints:

$$x(Q) = \sum_{i \in Q} x_i \leq 1$$

are valid inequalities for all cliques $Q \subseteq G$ and define facets iff Q is maximal

Clique constraint stable set polytope:
$\mathrm{QSTAB}(G) = \{x \in \mathbf{R}_+^{|G|} : x(Q) \leq 1 \text{ for } Q \subseteq G \text{ clique}\}$

Theorem (Chvátal 1975, Padberg 1974)

$\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ if and only if G is perfect.

Thus: Additional facets are required for any **imperfect** graph G since

$$\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$$

Goal: Consider appropriate generalizations of clique constraints, namely, rank constraints and clique family inequalities

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Rank constraints and rank-perfect graphs

Rank constraints:

$$x(G') = \sum_{i \in G'} x_i \leq \alpha(G')$$

are obviously valid inequalities for arbitrary induced subgraphs $G' \subseteq G$

Definition (W. 2000)

A graph G is rank-perfect iff $\text{STAB}(G) = \{x \in \mathbf{R}_+^{|G|} : x(G') \leq \alpha(G'),\ G' \subseteq G\}$.

Examples of rank-perfect graphs:

- perfect graphs
- t-perfect and h-perfect graphs (by definition)
- line graphs (Edmonds 1965)
- complements of webs and of fuzzy circular interval graphs (W. 2002, 2004)
- semi-line graphs (Chudnovsky & Seymour 2004)

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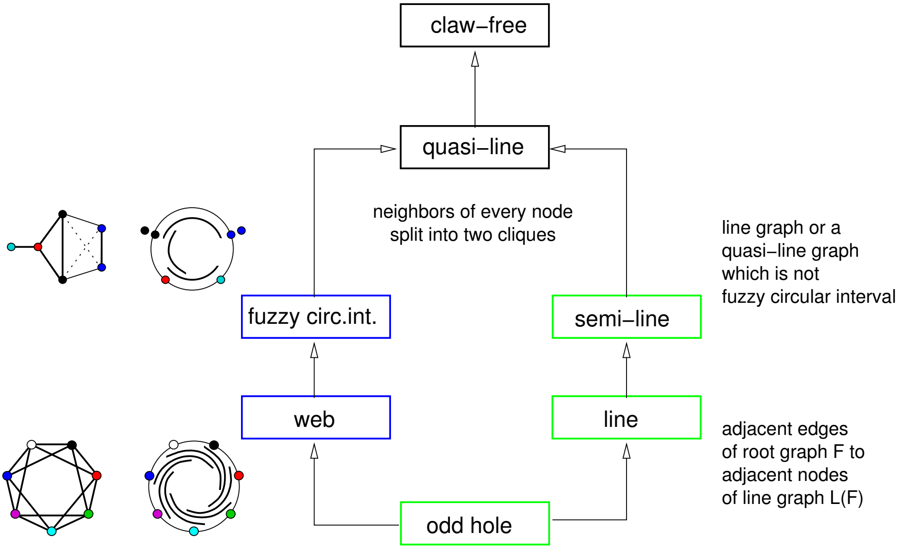
Definitions and inclusions of the studied graph classes



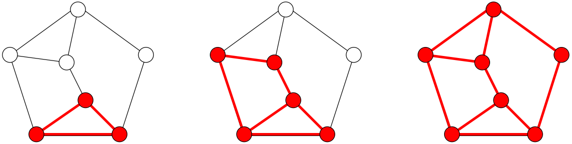
The rank facets of claw-free graphs

Theorem (Galluccio & Sassano 1997)

All rank facets of the stable set polytope of claw-free graphs can be obtained by means of standard techniques from

- **cliques**,
- **line graphs** of 2-connected hypomatchable graphs,
- partitionable **webs** $W_{\alpha\omega+1}^{\omega-1}$.

A graph H is **hypomatchable** if $H - v$ has a perfect matching for all nodes v.



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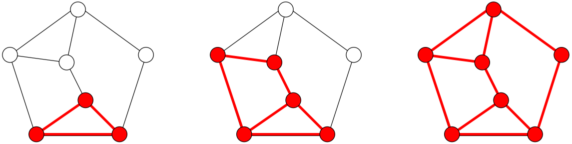

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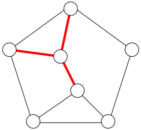
Edmonds' description of matching polytopes

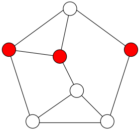
Theorem (Edmonds 1965)

The matching polytope $\mathrm{M}(G) = \mathrm{conv}\{\chi^M : M \subseteq E(G) \text{ matching}\}$ is given by

- trivial inequalities:
  $$x_e \geq 0 \ \forall \text{ edges } e \in E(G)$$
- edge star inequalities:
  $$x(\delta(v)) \leq 1 \ \forall v \in V(G),\ \delta(v) = \{e \in E(G) : e \text{ incident to } v\}$$
- odd set inequalities:
  $$x(E[H]) \leq \frac{|H|-1}{2} \ \forall H \subseteq V(G) \text{ with } |H| \geq 3 \text{ odd}$$

edge star





odd set

Theorem (Edmonds & Pulleyblank 1974)

An odd set inequality defines a **facet** if H is 2-connected, hypomatchable.

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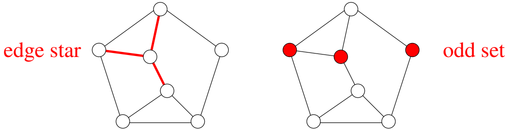

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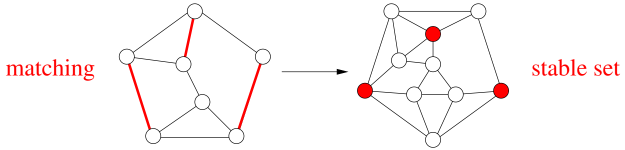
Consequences for stable set polytopes of line graphs

Line graph $L(F)$: (non)adjacent edges of F become (non)adjacent nodes of $L(F)$



Corollary

For any line graph $G = L(F)$, its stable set polytope STAB(G) is given by

- trivial inequalities:
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- clique inequalities:
  $$x(Q) \leq 1 \ \forall \text{cliques } Q \in G$$
- rank inequalities:
  $$x(L(H)) \leq \left\lfloor \frac{|H|}{2} \right\rfloor \ \forall H \subseteq F \text{ 2-connected, hypomatchable}$$

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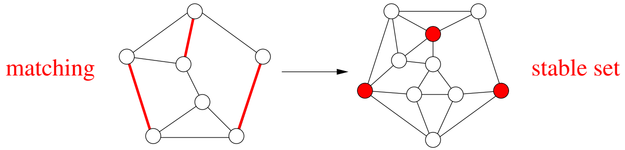

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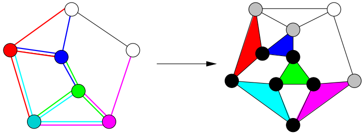
Extending odd set inequalities to clique family inequalities

odd set in F



clique family in L(F)

Definition: clique family inequality $(\mathcal{Q}, p)$ (CFI)

Let $\mathcal{Q}$ be a family of ≥ 3 maximal cliques, $p \leq |\mathcal{Q}|$ a parameter, and

$$I(\mathcal{Q}, p) = \{v \in V : |\{Q \in \mathcal{Q} : v \in Q\}| \geq p\}$$
$$O(\mathcal{Q}, p) = \{v \in V : |\{Q \in \mathcal{Q} : v \in Q\}| = p - 1\}$$

Then, for $r = |\mathcal{Q}| \bmod p$, $r > 0$, define the CFI $(\mathcal{Q}, p)$ as

$$(p - r) \sum_{v \in I(\mathcal{Q},p)} x_v + (p - r - 1) \sum_{v \in O(\mathcal{Q},p)} x_v \leq (p - r) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

Example: The CFI $(\mathcal{Q}, 2)$ of STAB(L(F)) is 1 x(●) + 0 x(◎) $\leq$ 2

The stable set polytope of quasi-line graphs

For which graphs do clique family inequalities suffice?

Ben Rebea Conjecture (1980)

The stable set polytope of any quasi-line graph is given by three types of constraints:

- nonnegativity constraints,
- clique constraints,
- clique family inequalities.

Conjecture verified for:

- line graphs (Edmonds 1965/Oriolo 2003)
- semi-line graphs (Chudnovsky and Seymour 2004)
- fuzzy circular interval graphs/quasi-line graphs (Eisenbrand, Oriolo, Stauffer, and Ventura 2005)

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Facet-defining clique family inequalities

Which clique family inequalities are **essential**?

- **line graphs:**
  $(\mathcal{Q}, 2)$ with $I(\mathcal{Q}, 2)$ line graph of a 2-connected hypomatchable graph
- **semi-line graphs:**
  clique family inequalities $(\mathcal{Q}, 2)$ with $|\mathcal{Q}|$ odd

Conjecture (Pêcher & W. 2004), Theorem (Stauffer 2005)

The stable set polytope of any **web** W_n^k admits only the following types of facets:

- nonnegativity constraints,
- clique constraints,
- full rank constraint $x(W_n^k) \leq \alpha(W_n^k)$,
- clique family inequalities $(\mathcal{Q}, k'+1)$ associated with proper subwebs $W_{n'}^{k'}$.

Conjecture extended to **fuzzy circular interval graphs** (Pêcher & W. 2006)

- if true: webs would be crucial for all rank and non-rank facets of fuzzy circular interval graphs

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The Chvátal-rank of inequalities and polytopes

Consider a polyhedron $P \subseteq \mathbb{R}^n$ and $P_I = \text{conv}\{x \in \mathbb{Z}^n : x \in P\}$.

For any valid inequality $\sum a_i x_i \leq b$ of P with $a_i \in \mathbb{Z}$, the inequality

$$\sum a_i x_i \leq \lfloor b \rfloor$$

is a **Chvátal-Gomory cut** for P and valid for P_I.

The set P' of points satisfying all such Chvátal-Gomory cuts for P is its **Chvátal-closure**. Let $P^{t+1} = (P^t)'$, then

$$P_I \subseteq P^t \subseteq P^0 = P$$

holds for every t.

Definition

- An inequality $\sum a_i x_i \leq b$ has Chvátal-rank at most t if it is valid for P^t.
- The smallest t with $P^t = P_I$ is the Chvátal-rank of P.

Odd set inequalities and the fractional matching polytope have Chvátal-rank 1.

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Conjecture (Edmonds 1973)

Claw-free graphs have Chvátal-rank 1.

The conjecture is **true** for line graphs (as odd set inequalities and, therefore, $P = \text{QSTAB}(G)$ for any line graph G have Chvátal-rank 1).

Counterexample (Giles & Trotter 1981, Oriolo 2003)

The fuzzy circular interval graph obtained by joining the webs W_{37}^{6} and W_{37}^{7} in a certain way has a clique family facet $(\mathcal{Q}, 8)$.
This clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank at least 2.

Thus, the conjecture is **not true** in general!

Problem

- Is the conjecture true for other classes of claw-free graphs?
- Is there an upper bound for the Chvátal-rank of quasi-line graphs?

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The Chvátal-rank of clique family inequalities

Theorem (Pêcher & W. 2005)

Let $(\mathcal{Q}, p)$ be a clique family inequality and let $r = |\mathcal{Q}| \pmod p$. For every $1 \leq i \leq p - r$, the inequality

$$i \sum_{v \in I(\mathcal{Q},p)} x_v + (i-1) \sum_{v \in O(\mathcal{Q},p)} x_v \leq i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

has Chvátal-rank at most i.

Remark: gives an alternative proof for the validity of clique family inequalities, involving only standard rounding arguments.

Corollary (Pêcher & W. 2005)

- A clique family inequality $(\mathcal{Q}, p)$ has Chvátal-rank at most $p - r$.
- Every rank clique family inequality has Chvátal-rank 1.

Consequence: Semi-line graphs have Chvátal-rank 1, thus Edmonds' conjecture is true for semi-line graphs.

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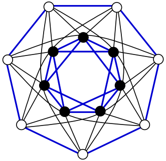
Chvátal-rank of clique family inequalities: Examples

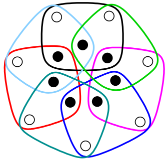
Example (Giles & Trotter 1981)

For any $k \geq 1$, the graph $G^k = W_n^{k+1} \times W_n^k$ has a clique family facet $(\mathcal{Q}, k+2)$

$$(k+1)x(W_n^{k+1}) + kx(W_n^k) \leq (k+1)\left\lfloor \frac{n}{k+2} \right\rfloor$$

where $\mathcal{Q}$ is of size $n = 2k(k+2)+1$.





Example (Liebling, Oriolo, Spille, and Stauffer 2004)

For any $a \geq 1$, the web $W_{(2a+3)^2}^{2(a+2)}$ has a clique family facet $(\mathcal{Q}, a+2)$

$$(a+1)x(I(\mathcal{Q}, a+2)) + ax(O(\mathcal{Q}, a+2)) \leq (a+1)\left\lfloor \frac{|\mathcal{Q}|}{a+2} \right\rfloor$$

where $\mathcal{Q}$ is of size $(a+2)(2a+3)$.

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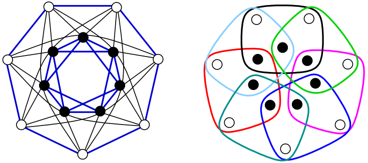

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$$(a+1)x(I(\mathcal{Q}, a+2)) + ax(O(\mathcal{Q}, a+2)) \leq (a+1)\left\lfloor \frac{|\mathcal{Q}|}{a+2} \right\rfloor$$

where $\mathcal{Q}$ is of size $(a+2)(2a+3)$.

Chvátal-rank of clique family inequalities: Improved bound

Theorem (Pêcher & W. 2005)

A clique family inequality $(\mathcal{Q}, p)$ with $r = |\mathcal{Q}| \pmod p$ has Chvátal-rank **at most**

$$\min\{r, p - r\}$$

Example: The above clique family inequalities with arbitrarily high coefficients have Chvátal-rank **one** as $r = 1$ holds in both cases.

Corollary (Pêcher & W. 2005)

A clique family inequality $(\mathcal{Q}, p)$ has Chvátal-rank at most $\frac{p}{2}$.

Consequence:

- All facets of a web W_n^k have Chvátal-rank at most $\frac{k-1}{2}$.
- There is no general upper bound on the Chvátal-rank, as for any $k \geq 1$, there are clique family facets $(\mathcal{Q}, 2k+1)$ with $k = \min\{2k+1-k, k\}$.

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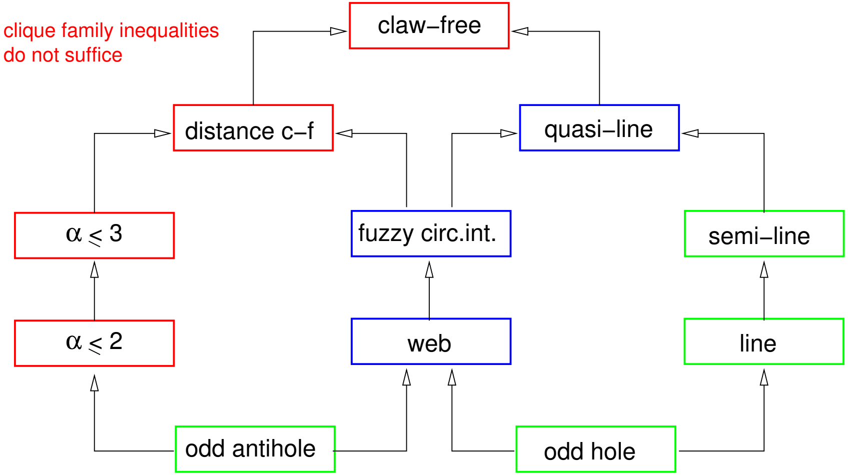
Outline

Beyond clique family inequalities and quasi-line graphs



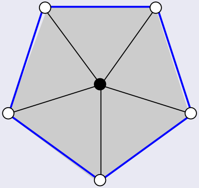
A graph is **distance claw-free** if, for every of its nodes v, neither $N(v)$ nor $N_2(v)$ contains a stable set of size 3.

More complex facets for general claw-free graphs

There are claw-free graphs whose stable set polytopes admit facets neither induced by cliques nor clique families:

Three Examples

5–wheel

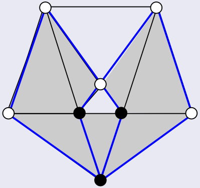


$x(\circ) + 2x(\bullet) \leq 2$

(Q,3) with r=2yields

$0\,x(\circ) + 1x(\bullet) \leq 1$

wedge

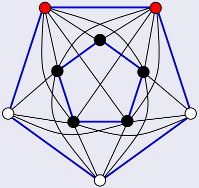


$x(\circ) + 2x(\bullet) \leq 3$

(Q,3) with r=1 yields

$x(\circ) + 2x(\bullet) \leq 4$

graph G_3



$x(\circ) + 2x(\bullet) + 3x(\bullet) \leq 4$

more than two non–zero coefficients required

The graphs with stability number two

Theorem (Cook 1987)

The stable set polytope of any graph G with $\alpha(G) \leq 2$ is entirely described by

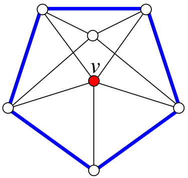
- trivial inequalities:

  $x_v \geq 0 \ \forall v \in V(G)$

- clique neighborhood inequalities $\mathrm{F}(Q)$:

  $2x(Q) + 1x(N'(Q)) \leq 2$ for all cliques Q where $N'(Q) = \{v : Q \subseteq N(v)\}$

and $\mathrm{F}(Q)$ is a facet iff $N'(Q)$ has in $\overline{G}$ no bipartite component.



Q	$N'(Q)$
maximal	$\emptyset$
$\{v\}$	C_5
$\emptyset$	$V(G)$

The graphs with stability number at least four

A connected claw-free graph G with $\alpha(G) \geq 4$

- is either fuzzy circular interval or can be composed from linear interval strips (Chudnovsky & Seymour 2005)
- **is quasi-line iff G does not contain a 5-wheel** (Fouquet 1993)
- has constraints associated with induced 5-wheels which can be lifted to more general inequalities $1x(\circ) + 2x(\bullet) \leq 2$ (Stauffer 2005)

Conjecture (Stauffer 2005)

The stable set polytope of a claw-free but not fuzzy circular interval graph G with $\alpha(G) \geq 4$ is given by

- nonnegativity constraints
- rank constraints
- lifted 5-wheel constraints

This would imply: all non-rank facets of a claw-free but not fuzzy circular interval graph G with $\alpha(G) \neq 3$ are **clique neighborhood constraints!**

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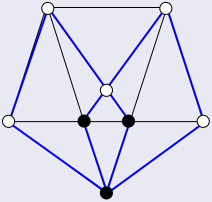
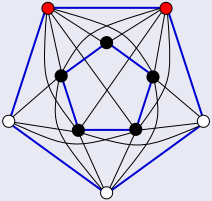
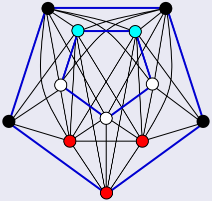
Examples (Giles & Trotter 1981, Liebling et al. 2004)

wedge	Giles & Trotter graph	fish in a net
		
x(○) + 2x(●) ≦ 3	x(○) + 2x(●) + 3x(● [red]) ≦ 4	x(○) + 2x(●) + 3x(● [red]) + 4x(● [cyan]) ≦ 5

Observation: all the known examples of complicated facets for claw-free graphs occur in the case $\alpha(G) = 3$, but they are not well-understood (so far)

Our goal: describe their structure!

The graphs with stability number three: Known Facets

Examples (Giles & Trotter 1981, Liebling et al. 2004)

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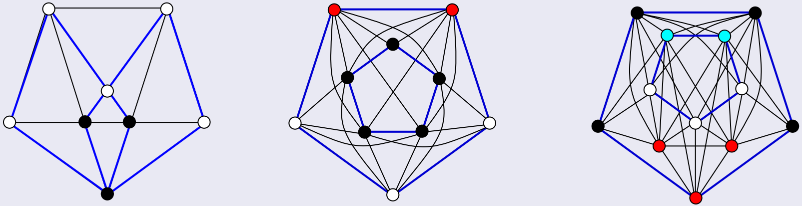

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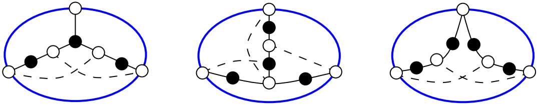
The graphs with stability number three: Wedges

A wedge is a claw-free graph G s.t. $\overline{G}$ has

- a unique triangle Δ
- a spanning tree T with 2 or 3 spokes of appropriate length
- additional edges (to avoid claws in G)



Theorem (Giles & Trotter 1981)

Every wedge induces the facet

$$1x(\circ) + 2x(\bullet) \leq 3$$

and its **roots** (= tight stable sets) correspond to the following cliques of $\overline{G}$:

- the $|G| - 1$ edges of the spanning tree T
- the unique triangle Δ

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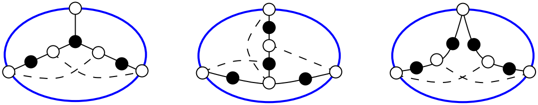

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Definition

Consider a graph G with $\alpha(G) = 3$. A non-rank facet $a^T x \leq b$ of $\mathrm{STAB}(G)$ is a co-spanning tree constraint if its roots correspond to the following cliques of $\overline{G}$:

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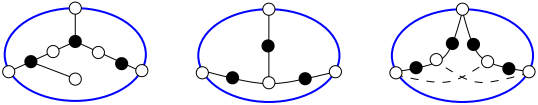

Observation

- the facets of wedges are of this type
- all such facets are of the form $1x(\circ) + 2x(\bullet) \leq 3$

Thus: generalize further to obtain more than two and higher coefficients!

1. Extension: Co-spanning tree constraints

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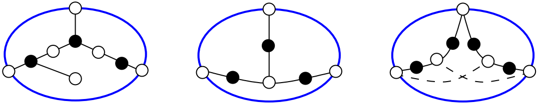

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2. Extension: Co-spanning forest constraints

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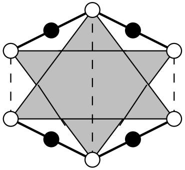
Consider a graph G with $\alpha(G) = 3$. A non-rank facet $a^T x \leq b$ of $\mathrm{STAB}(G)$ is a co-spanning forest constraint if its roots correspond to the following cliques of $\overline{G}$:

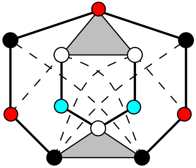
- the $|G| - k$ edges of a spanning **forest** F with k tree-components
- k triangles

new example



$x(\circ) + 2x(\bullet) \leq 3$

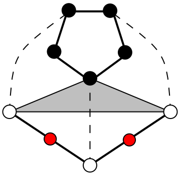
fish in a net



$x(\circ) + 2x(\bullet) + 3x(\text{red}) + 4x(\text{cyan}) \leq 5$

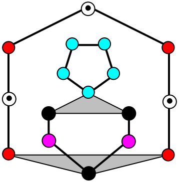
3. Extension: Co-spanning 1-forest constraints

Giles & Trotter graph



$x(\circ) + 2x(\bullet) + 3x(\bullet) \leq 4$

fish in a net with bubble



$2x(\bullet) + 3x(\bullet) + 4x(\bullet) + 5x(\odot) + 6x(\bullet) \leq 8$

Definition

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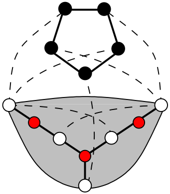
- the $|G| - k$ edges of a spanning 1-forest F consisting of some **odd 1-trees** and k **trees** as components
- k triangles

The graphs with stability number three: The Description

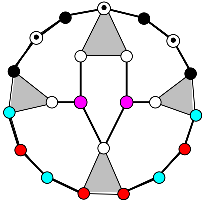
Theorem (Pêcher, W. 2006)

If $\alpha(G) = 3$, then all non-rank, non-complete join facets $a^T x \leq b$ are

- **co-spanning forest constraints** if b is odd;
- **co-spanning 1-forest constraints** if b is even.



2x(○) + 3x(●) + 4x(●) ≤ 6



x(○) + 2x(●) + 3x(●) + 4x(●) + 5x(⊙) + 6x(●) ≤ 7

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In the stable set polytope of a claw-free graph G with $\alpha(G) \leq 3$, every non-rank facet is a **co-spanning 1-forest constraint**.

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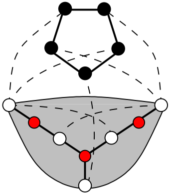
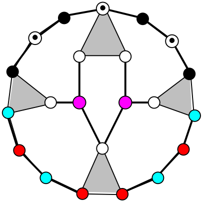


2x(○) + 3x(●) + 4x(red ●) ≤ 6



x(○) + 2x(●) + 3x(red ●) + 4x(cyan ●) + 5x(⊙) + 6x(magenta ●) ≤ 7

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Outline

The types of non-rank facets for claw-free graphs

Combine the results/conjectures on non-rank facets for claw-free graphs G with

- $\alpha(G) = 2$ (Cook 1987)
- $\alpha(G) = 3$ (Pêcher, W. 2006)
- $\alpha(G) \geq 4$ (Stauffer 2005)

Conjecture (Pêcher, W. 2006)

A non-rank facet associated with a claw-free graph G is a

- **clique neighborhood constraint** if $\alpha(G) = 2$
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- **clique family inequality** or a **clique neighborhood constraint** if $\alpha(G) \geq 4$

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Types of facet-defining subgraphs

Conjecture (Pêcher, W. 2006)

All non-rank facets of the stable set polytope of claw-free graphs rely on

- **odd antiwheels** (clique neighborhood constraints),
- **co-spanning 1-forests** (co-spanning 1-forest constraints),
- **prime webs** (clique family inequalities).

Conjecture (Pêcher & W. 2006)

for non-clique facets of the stable set polytope of quasi-line graphs: