Two-Period Convex Hull Closures for Big Bucket Lot-Sizing Problems

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Outline

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Lot-Sizing: What's this about?

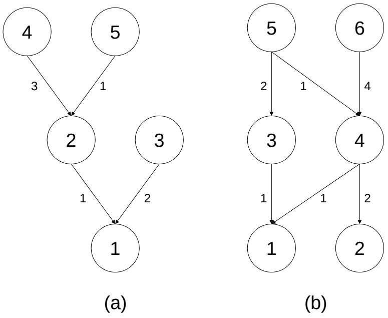
- "Items" to manufacture
- "Demands" to be satisfied
  - Forecasting (e.g., Peugeot)
  - Make-to-order (e.g., Airbus)
- "System limitations" such as capacities
- Decisions to be made each period
  - To produce or not to produce?
  - How much to produce?
  - How much to stock?
  - ...
  - **Decision factor:** Costs/revenues

Motivation for Lot-Sizing

- $$$!! Highly competitive markets for manufacturing companies
  - Significant area for cost improvement
  - Current automated systems even short of ensuring feasibility
- Lot-Sizing problems of realistic size/complexity too difficult for MIP solvers
  - Usually no room for expectation of optimality!
- Current polyhedral techniques usually limited to extensions of single-item techniques
  - Simply too naive to provide a thorough understanding of complicated problems
- **Question**: What can we do to obtain better lower bounds?

Problem Description

- Multiple items and levels (BOM structure)
  - Assembly (a) or general (b) structures



- Demands
- Big-bucket capacities (items share resources)
- Extensions possible, e.g. overtime and backlogging
- Production plan minimizing total cost to be determined

Problem Characteristics

- Decision variables (in each time period t for each item i)
  - Production setup decisions (y_t^i)
  - Production amounts (x_t^i)
  - Inventory held (s_t^i)
- Constraints
  - Flow conservation/demand satisfaction
    - Internal/external demand
  - Capacity limits (big bucket)
  - Setup-production relations

Basic Formulation

$$\min \sum_{t=1}^{NT}\sum_{i=1}^{NI} f_t^i y_t^i + \sum_{t=1}^{NT}\sum_{i=1}^{NI} h_t^i s_t^i \tag{1}$$

$$\text{s.t. } x_t^i + s_{t-1}^i - s_t^i = d_t^i \qquad t \in [1, NT], i \in endp \tag{2}$$

$$x_t^i + s_{t-1}^i - s_t^i = \sum_{j \in \delta(i)} r^{ij} x_t^j \qquad t \in [1, NT], i \notin endp \tag{3}$$

$$\sum_{i=1}^{NI} (a_k^i x_t^i + ST_k^i y_t^i) \leq C_t^k \qquad t \in [1, NT], k \in [1, NK] \tag{4}$$

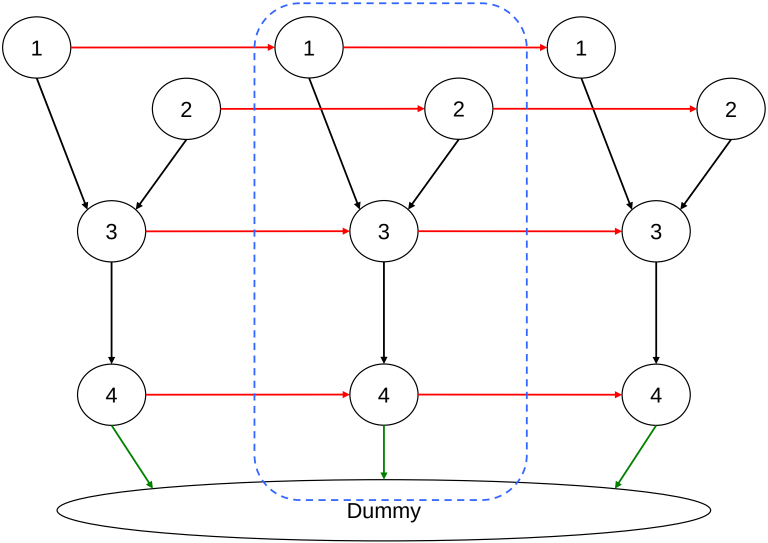
$$x_t^i \leq M_t^i y_t^i \qquad t \in [1, NT], i \in [1, NI] \tag{5}$$

$$y \in \{0, 1\}^{NTxNI} \tag{6}$$

$$x \geq 0 \tag{7}$$

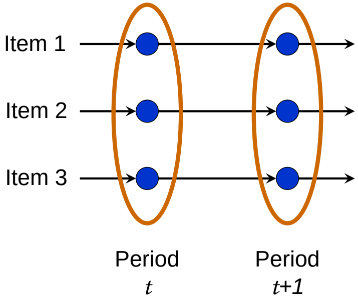
$$s \geq 0 \tag{8}$$

As a Fixed Charge Network



What do we know?

- Many of the test problems are challenging
- We do not have an adequate approximation of the convex hull of the **multi-item, single-machine, single-level capacitated problems**! (Akartunalı and Miller [2007])



- Generalizing the “bottleneck flow” model of Atamtürk and Muñoz [2004]

The Model to Study

$$x^i_{t'} \leq \widetilde{M}^i_{t'} y^i_{t'} \qquad i = [1, ..., NI], t' = 1, 2 \qquad (9)$$

$$x^i_{t'} \leq \widetilde{d}^i_{t'} y^i_{t'} + s^i \qquad i = [1, ..., NI], t' = 1, 2 \qquad (10)$$

$$x^i_1 + x^i_2 \leq \widetilde{d}^i_1 y^i_1 + \widetilde{d}^i_2 y^i_2 + s^i \qquad i = [1, ..., NI] \qquad (11)$$

$$x^i_1 + x^i_2 \leq \widetilde{d}^i_1 + s^i \qquad i = [1, ..., NI] \qquad (12)$$

$$\sum_{i=1}^{NI} (a^i x^i_{t'} + ST^i y^i_{t'}) \leq \widetilde{C}_{t'} \qquad t' = 1, 2 \qquad (13)$$

$$x, s \geq 0, y \in \{0, 1\}^{2xNI} \qquad (14)$$

- Let $X^{2PL} = \{(x, y, s) | (9) - (14)\}$

Outline

Two-Period Relaxation: Basics

- *Motivation 1:* Two-period problems are computationally easy to solve
  - Our experience from heuristic frameworks
- *Motivation 2:* There are recent promising results on closures
  - E.g. on MIR and Split Closures (Andersen, Cornuejols, Dash, Günlük, Lodi, ...)
- *Motivation 3:* 1-period problems are not strong enough
  - Miller, Nemhauser and Savelsbergh [2000], [2003]; Jans and Degraeve [2004]
- **Basic idea:** Generating cuts to separate an LPR solution over the convex hull of the two-period problems
  - *Advantage:* No need for information about the structure of two-period problems
  - *Caution:* But we want to understand the structure of two-period problems

Separation Over the Two-Period Convex Hull

LPR of the original problem $\Rightarrow$ A solution $(\tilde{x}, \tilde{y}, \tilde{s})$
L^1 (Manhattan distance) problem:

$$z^1 = \min_{\Delta,\lambda} \sum_i [(\Delta_s^-)^i + \sum_{t'=1}^{2} (\Delta_x^+)^i_{t'} + (\Delta_x^-)^i_{t'} + (\Delta_y^+)^i_{t'} + (\Delta_y^-)^i_{t'}]$$

$$\begin{aligned}
\text{s.t. } \tilde{x}^i_{t'} &= \sum_k \lambda_k (x_k)^i_{t'} + (\Delta_x^+)^i_{t'} - (\Delta_x^-)^i_{t'} && \forall i, t' = 1,2 && (\alpha^i_{t'}) \\
\tilde{y}^i_{t'} &= \sum_k \lambda_k (y_k)^i_{t'} + (\Delta_y^+)^i_{t'} - (\Delta_y^-)^i_{t'} && \forall i, t' = 1,2 && (\beta^i_{t'}) \\
\tilde{s}^i &\geq \sum_k \lambda_k (s_k)^i - (\Delta_s^-)^i && \forall i && (\gamma^i) \\
& \sum_k \lambda_k \leq 1 && && (\eta) \\
& \lambda_k \geq 0, \ \Delta \geq 0
\end{aligned}$$

Separation Over the Two-Period Convex Hull (cont'd)

The dual of the L^1 problem:

$$\max_{\alpha,\beta,\gamma,\eta} \sum_{i=1}^{NI} \sum_{t'=1}^{2} (\tilde{x}^i_{t'} \alpha^i_{t'} + \tilde{y}^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} \tilde{s_k}^i \gamma^i + \eta \tag{15}$$

$$\text{s.t.} \sum_{i=1}^{NI} \sum_{t'=1}^{2} ((x_k)^i_{t'} \alpha^i_{t'} + (y_k)^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} (s_k)^i \gamma^i + \eta \leq 0 \quad \forall k \tag{16}$$

$$-1 \leq \alpha^i_{t'} \leq 1 \quad \forall i, t' \tag{17}$$

$$-1 \leq \beta^i_{t'} \leq 1 \quad \forall i, t' \tag{18}$$

$$-1 \leq \gamma^i \leq 0 \quad \forall i \tag{19}$$

$$\eta \leq 0 \tag{20}$$

Separation Over the Two-Period Convex Hull (cont'd)

Theorem

Let $z^1 > 0$ for $(\tilde{x}, \tilde{y}, \tilde{s})$, and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$ be optimal dual values. Then,

$$\sum_{i=1}^{NI} \sum_{t'=1}^{2} (\bar{\alpha}^i_{t'} x^i_{t'} + \bar{\beta}^i_{t'} y^i_{t'}) + \sum_i \bar{\gamma}^i s^i + \bar{\eta} \leq 0 \qquad (21)$$

is a valid inequality for $conv(X^{2PL})$ that cuts off $(\tilde{x}, \tilde{y}, \tilde{s})$.

Proof.

Validity: Using (16), $\bar{\gamma} \leq 0$ and $\lambda \geq 0$.
Violation for $(\tilde{x}, \tilde{y}, \tilde{s})$: Using (15) □

Generating Extreme Points for Separation

- How to generate (x_k, y_k, s_k)?
  - Using column generation
  - Solve the pricing problem using the optimal $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$:

$$\max_{x,y,s} \; z_P = \sum_i \sum_{t'=1}^{2} (\bar{\alpha}^i_{t'} x^i_{t'} + \bar{\beta}^i_{t'} y^i_{t'}) + \sum_i \bar{\gamma}^i s^i + \bar{\eta}$$
$$\text{s.t. } (x, y, s) \in X^{2PL}$$

  - If $z_P > 0$, then the solution is an extreme point of X^{2PL}; otherwise, generating extreme points is done

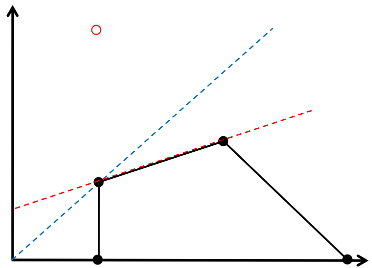
Separation Algorithm

repeat
  Solve the distance problem for $conv(X^{2PL})$
  if $z^1 = 0$ **then** break
    else Solve column generation problem
    if $z_P \leq 0$ **then** break
      else Add new extreme point

until $z^1 = 0$ **or** $z_P \leq 0$
if $z^1 = 0$ **then** $(\tilde{x}, \tilde{y}, \tilde{s}) \in conv(X^{2PL})$
  else Add the violated cut (21)

Separation Over Two-Period Convex Hull: Other Norms

- Different norms
  - $\Rightarrow$ Different convergence
  - $\Rightarrow$ Different cuts
- Example: $K = \{(1,0),(1,1),(2.5,1.5),(4,0)\}$, $\tilde{x} = (1,3)$.
  - L^1: Distance $z_1 = 2$; cut $-x_1 + x_2 \leq 0$.
  - L^∞: Distance $z_\infty = 1.5$; cut $-0.25x_1 + 0.75x_2 - 0.5 \leq 0$.
  - L^2: Distance $z_2 = \sqrt{3.6}$; cut $1.2x_1 - 3.6x_2 + 2.4 \geq 0$.



Separation Over Two-Period Convex Hull: Other Norms

- L^∞ almost identical
  - Still a linear model
    - Similar to L^1 problem
    - Fewer variables
  - Methodology and theory remains almost identical
- L^2 (Euclidean distance) trickier
  - PSD matrix $\Rightarrow$ use of QP strong duality
  - Details to follow ...
  - *Remark:* Current QP solvers are almost as fast as LP solvers
- Combinations of different norms?
  - More on this in computational results ...

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Using Euclidean Distance (L^2)

$$z^2 = \min_{\Delta,\lambda} \sum_i \left[[(\Delta_s)^i]^2 + \sum_{t'=1}^{2} \left([(\Delta_x)^i_{t'}]^2 + [(\Delta_y)^i_{t'}]^2 \right) \right]$$

$$\text{s.t. } \tilde{x}^i_{t'} = \sum_k \lambda_k (x_k)^i_{t'} + (\Delta_x)^i_{t'} \qquad \forall i, t' = 1,2 \qquad (\alpha^i_{t'})$$

$$\tilde{y}^i_{t'} = \sum_k \lambda_k (y_k)^i_{t'} + (\Delta_y)^i_{t'} \qquad \forall i, t' = 1,2 \qquad (\beta^i_{t'})$$

$$\tilde{s}^i \geq \sum_k \lambda_k (s_k)^i - (\Delta_s)^i \qquad \forall i \qquad (\gamma^i)$$

$$\sum_k \lambda_k \leq 1 \qquad (\eta)$$

$$\lambda_k \geq 0, \;\; \Delta_s \geq 0, \;\; \Delta_x, \Delta_y \text{ free}$$

Using Euclidean Distance: Dual

$$z_D = \max_{\Delta,\alpha,\beta,\gamma} -\sum_i \left[[(\Delta_s)^i]^2 + \sum_{t'=1}^{2} [(\Delta_x)^i_{t'}]^2 + [(\Delta_y)^i_{t'}]^2 \right]$$

$$-\left(\sum_{i=1}^{NI} \sum_{t'=1}^{2} (\tilde{x}^i_{t'} \alpha^i_{t'} + \tilde{y}^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} \tilde{s}^i \gamma^i + \eta \right)$$

$$\text{s.t.} \sum_{i=1}^{NI} \sum_{t'=1}^{2} ((x_k)^i_{t'} \alpha^i_{t'} + (y_k)^i_{t'} \beta^i_{t'}) + \sum_{i=1}^{NI} (s_k)^i \gamma^i + \eta \geq 0 \quad \forall k$$

$$\alpha^i_{t'} = -2(\Delta_x)^i_{t'} \quad \forall i, t'$$

$$\beta^i_{t'} = -2(\Delta_y)^i_{t'} \quad \forall i, t'$$

$$-\gamma^i \geq -2(\Delta_s)^i \quad \forall i$$

$$\gamma \geq 0,\ \eta \geq 0,\ \Delta_s \geq 0,\ \alpha, \beta, \Delta_x, \Delta_y \text{ free}$$

Using Euclidean Distance: Theory

Theorem

Let $z^2 > 0$ for $(\tilde{x}, \tilde{y}, \tilde{s})$, with optimal primal values $(\bar{\Delta}_x, \bar{\Delta}_y, \bar{\Delta}_s, \bar{\lambda})$, and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$ be the associated optimal dual values. Then,

$$\sum_i \sum_{t'=1}^{2} (\bar{\alpha}^i_{t'} x^i_{t'} + \bar{\beta}^i_{t'} y^i_{t'}) + \sum_i \bar{\gamma}^i s^i + \bar{\eta} \geq 0 \qquad (22)$$

is a valid inequality for $conv(X^{2PL})$ that cuts off $(\tilde{x}, \tilde{y}, \tilde{s})$.

Proof.

Using a similar approach to the previous proof and also using the strong duality theorem for QP. □

Defining Two-Period Subproblems

- **Question 1:** On which two periods to run the separation?
  - We can look at all the two-period problems ($NT - 1$ of them)
- **Question 2:** Which period's stock is represented by s^i?
  - Let $\phi(i) \in [t+1, .., NT]$ be the horizon parameter for each i
  - Obvious choice: $t+1$, i.e., $s^i = s^i_{t+1}$
  - Then, parameters are defined as follows ($\forall i,\ t' = 1, 2$):
    - $\widetilde{M}^i_{t'} = M^i_{t+t'-1}$, $\widetilde{C}^i_{t'} = C^i_{t+t'-1}$
    - $\widetilde{d}^i_{t'} = d^i_{t+t'-1,\ t+1}$, i.e., $\widetilde{d}^i_1 = d^i_{12}$ and $\widetilde{d}^i_2 = d^i_2$.
- **Observation 1:** If a number of periods following $t+1$ have no setups, their demands should be incorporated
- **Observation 2:** If a setup occurs after $t+1$, (ℓ, S) inequalities will be weakened if that period's demand is in

Two-Period Convex Hull Closure Framework

- Following Miller, Nemhauser, Savelsbergh (2000)

$$\phi(i) = \max\{t' | t' \geq t+1, \sum_{t''=t+1}^{t'} y^i_{t''} \leq y^i_{t+1} + \Theta\}$$

  where $\Theta \in (0,1]$ is a random number
- Let X^{2PL}_t be $X^{2PL}_t(\phi(1), \phi(2), ..., \phi(NI))$

  Solve LPR of the original problem
  $\rightarrow (\tilde{x}, \tilde{y}, \tilde{s})$
  for t=1 **to** NT-1
    Define X^{2PL}_t
    Apply two-period convex hull separation algorithm

Outline

Computational Results: Two-Period Problems

- *2PCLS* instances: 20 problems with two periods only and with two to six items
  - **cdd** might provide the full description of the convex hull
    - Generate all the extreme points and rays of the LPR
    - Eliminate all the fractional extreme points
    - Using these integral extreme points, generate all the facets of the integral polyhedron
  - The more items share a resource, the more the structure tends to resemble that of an uncapacitated problem
- The separation algorithm implemented in Mosel (Mosel version 2.4.0, Xpress 2008A package)
- XLP based on a strong formulation using all violated (ℓ, S) inequalities (Barany et al. [1984])

Computational Results: Two-Period Problems (cont'd)

Instance	NI	XLP	IP	# cuts (L^1)	# cuts (L^∞)	# cuts (L^2)
2pcls01	3	17.033	25	11	8	17
2pcls02	3	12.6253	19	13	7	7
2pcls03	3	76.5345	104	5	3	1
2pcls04	2	14.7674	19	4	2	1
2pcls05	3	38.39	52	8	6	4
2pcls06	3	117.375	173	5	6	5
2pcls07	2	36.5	43	2	1	1
2pcls08	2	21.45	26	7	2	2
2pcls09	2	129	153	2	3	3
2pcls10	3	17.6539	24	1	3	1

Computational Results: Two-Period Problems (cont'd)

Instance	NI	XLP	IP	# cuts (L^1)	# cuts (L^∞)	# cuts (L^2)
2pcls11	3	71.7209	102	4	1	1
2pcls12	3	46.68	69	4	1	2
2pcls13	4	85.6256	113	7	7	9
2pcls14	4	70.2961	81	6	8	5
2pcls15	4	54.1848	74	6	3	1
2pcls16	4	34.0844	39	6	7	4
2pcls17	5	164.858	211	39	19	14
2pcls18	5	57.0825	97	34	10	6
2pcls19	6	115.131	150	11	6	1
2pcls20	6	59.2412	89	34	11	5

Computational Results: Two-Period Problems (cont'd)

Instance	L^∞		L^2	
	# cuts	# cols/ite	# cuts	# cols/ite
2pcls01	8	41.44	17	27.06
2pcls02	7	43.88	7	24.38
2pcls03	3	26.25	1	14
2pcls04	2	10	1	25
2pcls05	6	43.14	4	20.8
2pcls06	6	43.71	5	20.17
2pcls07	1	15	1	7
2pcls08	2	17.67	2	8.67
2pcls09	3	19.5	3	8.75
2pcls10	3	27.5	1	14
2pcls13	7	72.75	9	35.27
2pcls17	19	135.35	14	63.73
2pcls20	11	137.83	5	64.5

Computational Results: Multi-Period Problems

tr6-15 detailed results (30 iterations):

	L^1		L^2		
	lim=100 $\phi(i)$	lim=150 $\phi(i)$	lim=100 $t+1$	lim=150 $t+1$	lim=150 $\phi(i)$
2PL	37,234.6	37,298.8	37,306.7	37,306.8	37,331.2

Trigeiro instances results:

	tr6-15	tr6-30	tr12-15	tr12-30
XLP	37,201	60,946	73,848	130,177
2PL	37,364	61,096	73,962	130,350
OPT	37,721	61,746	74,634	130,596
Gap closed	31.35%	18.75%	14.50%	41.29%

Computational Results: Multi-Period Problems (cont'd)

TDS instances preliminary results with L^∞ approach:

	BK511131	BK511141	BK521131	BK521142
XLP	92,602	125,307	92,350	124,988
2PL	117,540	148,936	115,071	139,118
Best Sol.	120,303	162,629	118,217	153,805
Gap was	29.91%	29.78%	28.01%	23.06%
Gap now	2.35%	9.19%	2.73%	10.56%
	BK512131	BK512132	BK521132	BK522142
XLP	90,733	90,814	94,257	119,559
2PL	110,125	110,546	114,676	133,461
Best Sol.	113,536	112,809	117,423	148,471
Gap was	25.13%	24.22%	24.58%	24.18%
Gap now	3.10%	2.05%	2.40%	11.25%

Outline

Two-Period Model: Assumptions

- This section is far from complete!
- **Assumptions:**
  - $0 < \widetilde{M}_t^i, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\}$
    - Otherwise $x_t^i = 0$
  - $ST^i < \widetilde{C}_t, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\}$
    - Otherwise $y_t^i = 0, x_t^i = 0$

Proposition

$conv(X^{2PL})$ is full-dimensional.

Two-Period Model: Trivial Facets

Proposition

The following inequalities are facet-defining for $conv(X^{2PL})$*:*

1. $x_t^i \geq 0,\ t \in \{1,2\}, i \in \{1,\dots,NI\}$
2. $y_t^i \leq 1,\ t \in \{1,2\}, i \in \{1,\dots,NI\}$ **if** $ST^{i'} < \widetilde{C}_t - ST^i$ $\forall i' \in \{1,\dots,NI\} \setminus \{i\}$
3. $x_t^i \leq \widetilde{d}_t^i y_t^i + s^i$, $t \in \{1,2\}, i \in \{1,\dots,NI\}$ **if** $a^i \widetilde{d}_t^i + ST^i \leq \widetilde{C}_t$

Two-Period Model: Less-Trivial Facets

- More to discover - still in progress
  - Usual suspects: Inequalities based on covers (Padberg et al. (1984), Goemans (1989), Miller et al. (2002), Atamtürk and Muñoz (2004), ...)

Proposition

$$x_1^i + x_2^i \leq \widetilde{d}_1^i y_1^i + \widetilde{d}_2^i + s^i, \forall i \in \{1, \ldots, NI\}$$

is facet-defining **if** $a^i \widetilde{d}_t^i + ST^i < \widetilde{C}_t \ \forall t \in \{1, 2\}$.

Outline

Conclusions

- Study of a stronger relaxation
  - A new framework independent from defining families of valid inequalities or reformulations a priori, although expected output is to define new valid inequalities using the results from the framework
  - To our knowledge, this is an original approach in lot-sizing literature
- Different norms useful to generate cuts and improve lower bounds significantly
  - Euclidean and L^∞ approaches computationally much more efficient than Manhattan approach
    - Observed both on the efficiency of cuts and on the number of extreme points generated in column generation before termination

Ongoing Work

- Completing computational results on realistic size problems
  - Resolve computational issues
- Polyhedral analysis of the two-period relaxation
  - Careful analysis of the inequalities generated by the framework and the facets from **cdd**
  - Significant potential to identify new families of valid inequalities